Blowing up Solutions for a Biharmonic Equation with Critical Nonlinearity

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Abstract. In this paper we consider the following biharmonic equation with critical exponent (P_ε) : $\Delta^2 u = Ku^{\frac{n+4}{n-4}-\varepsilon}$, $u > 0$ in Ω and $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, ε is a small positive parameter, and K is a smooth positive function in $\overline{\Omega}$. We construct solutions of (P_ε) which blow up and concentrate at strict local maximum of K either at the boundary or in the interior of Ω. We also construct solutions of (P_ε) concentrating at an interior strict local minimum point of K. Finally, we prove a nonexistence result for the correponding supercritical problem which is in sharp contrast to what happened for (P_ε).

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1 Introduction and Results

In this paper, we are concerned with the concentration phenomena of the following biharmonic equation under the Navier boundary condition

$$(P_\varepsilon) \quad \begin{cases} \Delta^2 u = Ku^{p-\varepsilon}, \ u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, ε is a small positive parameter, $p+1 = 2n/(n-4)$ is the critical Sobolev exponent of the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and K is a smooth positive function in $\overline{\Omega}$.

The study of concentration phenomena for second order elliptic equations involving nearly critical exponent has attracted considerable attention in the last decades. See for example [1], [4], [7], [10], [11], [13], [14], [16], [17], [19], [20], [21], [22], [23] and the references therein.
However, as far as the authors know, the concentration phenomena for problem (P_ε) have been studied only in [12], [15] and [6] for $K \equiv 1$ only.

The purpose of the present paper is to construct solutions for (P_ε) concentrating at various point of Ω. More precisely, we are interested in constructing solutions concentrating at a strict local maximum point of K either at the boundary or in the interior of Ω. We will also construct solutions concentrating at an interior strict local minimum point of K. Similar results for Laplacian equation involving nearly critical Sobolev exponent has been proved by Chabrowski

and Yan [11]. Compared with the second order case, further technical difficulties have to be solved by means of delicate and careful estimates. Our method uses some techniques developed by Bahri [2], Rey[21] and Ben Ayed-El Mehdi [6] in the framework of *Theory of critical points at infinity*. The main idea consists in performing refined expansions of the Euler functional associated to our variational problem, and its gradient in a neighborhood of potential concentration sets. Such expansions are made possible through a finite dimension reduction argument.

To state our results, we need to introduce some notation. We denote by G the Green's function of Δ^2, that is,

$$\forall x \in \Omega \quad \begin{cases} \Delta^2 G(x,.) = c_n \delta_x & \text{in } \Omega \\ \Delta G(x,.) = G(x,.) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_x denotes the Dirac mass at x and $c_n = (n-4)(n-2)|S^{n-1}|$. We also denote by H the regular part of G, that is,

$$H(x,y) = |x-y|^{4-n} - G(x,y), \text{ for } (x,y) \in \Omega \times \Omega.$$

Let

$$\delta_{x,\lambda}(y) = \frac{c_0 \lambda^{\frac{n-4}{2}}}{(1+\lambda^2|y-x|^2)^{\frac{n-4}{2}}}, \ c_0 = [(n-4)(n-2)n(n+2)]^{(n-4)/8}, \ \lambda > 0, \ x \in \mathbb{R}^n. \tag{1.1}$$

It is well known (see [18]) that $\delta_{x,\lambda}$ are the only solutions of

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0 \text{ in } \mathbb{R}^n, \quad \text{with } u \in L^{p+1}(\mathbb{R}^n) \quad \text{and } \Delta u \in L^2(\mathbb{R}^n)$$

and are also the only minimizers of the Sobolev inequality on the whole space, that is

$$S = \inf\{||\Delta u||^2_{L^2(\mathbb{R}^n)} ||u||^{-2}_{L^{\frac{2n}{n-4}}(\mathbb{R}^n)}, \ s.t. \ \Delta u \in L^2, u \in L^{\frac{2n}{n-4}}, u \neq 0\}. \tag{1.2}$$

We denote by $P\delta_{x,\lambda}$ the projection of the $\delta_{x,\lambda}$'s onto $H^2(\Omega) \cap H^1_0(\Omega)$, defined by

$$\Delta^2 P\delta_{x,\lambda} = \Delta^2 \delta_{x,\lambda} \text{ in } \Omega \text{ and } \Delta P\delta_{x,\lambda} = P\delta_{x,\lambda} = 0 \text{ on } \partial\Omega$$

and we set

$$\varphi_{x,\lambda} = \delta_{x,\lambda} - P\delta_{x,\lambda}.$$

The space $\mathcal{H}(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$ is equipped with the norm $||.||$ and its corresponding inner product $(.,.)$ defined by

$$||u|| = \left(\int_\Omega |\Delta u|^2\right)^{1/2}, \qquad u \in \mathcal{H}(\Omega) \tag{1.3}$$

$$(u,v) = \int_\Omega \Delta u \Delta v, \qquad u, v \in \mathcal{H}(\Omega). \tag{1.4}$$

Let

$$|u|_q = |u|_{L^q(\Omega)} \tag{1.5}$$

$$E_{x,\lambda} = \{v \in \mathcal{H}(\Omega) / (v, P\delta_{x,\lambda}) = (v, \frac{\partial P\delta_{x,\lambda}}{\partial \lambda}) = (v, \frac{\partial P\delta_{x,\lambda}}{\partial x_j}) = 0, \ j = 1, ..., n\}. \tag{1.6}$$

Now we state the main results of this paper.

Theorem 1.1 *Let $x_0 \in \partial\Omega$ be a strict local maximum point of K satisfying*

$$K(x) \le K(x_0) - a|x - x_0|^{2+\alpha} \quad \forall x \in B_\mu(x_0) \cap \overline{\Omega}, \tag{1.7}$$

where $\mu > 0$, $a > 0$ and $\alpha \ge 0$ if $n \le 6$, $\alpha \in [0, 4/(n-6))$ if $n \ge 7$. Then there is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (P_ε) has a solution of the form

$$u_\varepsilon = \alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon, \tag{1.8}$$

where $v_\varepsilon \in E_{x_\varepsilon,\lambda_\varepsilon}$, and as $\varepsilon \to 0$,

$$\alpha_\varepsilon \to K(x_0)^{(4-n)/8},\ ||v_\varepsilon|| \to 0,\ x_\varepsilon \to x_0,\ \lambda_\varepsilon \to +\infty,\ \lambda_\varepsilon d(x_\varepsilon, \partial\Omega) \to +\infty. \tag{1.9}$$

Theorem 1.2 *Let $x_0 \in \Omega$ be a strict local maximum point of K. Then there is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (P_ε) has a solution of the form* (1.8) *satisfying* (1.9).

The aim of the next result is to show that if K is flat enough around a strict local minimum, (P_ε) has a solution concentrating at this point.

Theorem 1.3 *Let $x_0 \in \Omega$ be a strict local minimum point of K satisfying*

$$|D^l K(x)| \le C|x - x_0|^{L-l}, \quad l = 1, ..., n-4, \quad \forall x \in B_\mu(x_0), \tag{1.10}$$

$$|K(x) - K(x_0)| \ge C_0|x - x_0|^L, \quad \forall x \in B_\mu(x_0), \tag{1.11}$$

where $L > n-4$ is a constant, and where C and C_0 are positive constants.
Then there is an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$, (P_ε) has a solution of the form (1.8) *satisfying* (1.9) *and $\varepsilon\lambda_\varepsilon^{n-4} \to c > 0$.*

In the case $n = 5$ or 6, we can obtain a better result.

Theorem 1.4 *Assume that $x_0 \in \Omega$ is a strict local minimum point of K. If one of the following conditions is satisfied :*
(i) $n = 5$;
(ii) $n = 6$ *and*

$$c_1 H(x_0, x_0) - \frac{c_2 \Delta K(x_0)}{36 K(x_0)} > 0, \ \text{with}\ c_1 = c_0^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} \frac{dy}{(1 + |y|^2)^{\frac{n+4}{2}}}\, c_2 = \int_{\mathbb{R}^n} |y|^2 \delta_{o,1}^{p+1} dy, \tag{1.12}$$

then the conclusion of Theorem 1.3 holds.

The condition (1.12) is nearly necessary. Indeed, we have the following result:

Theorem 1.5 *Assume that $x_0 \in \Omega$ is a critical point of K satisfying one of the following conditions :*
(i) $n \ge 7$ *and* $\Delta K(x_0) > 0$,
(ii) $n = 6$ *and* $c_1 H(x_0, x_0) - \frac{c_2 \Delta K(x_0)}{36 K(x_0)} < 0$, *where c_1 and c_2 are the constants defined in Theorem 1.4.*
Then (P_ε) has no solution of the form (1.8) *satisfying* (1.9).

In contrast with the above results, we have the following nonexistence result for the supercritical problem.

Theorem 1.6 *Assume that $x_0 \in \Omega$ is a critical point of K satisfying one of the following conditions :*
(i) $n = 5$,
(ii) $n = 6$ *and* $c_1 H(x_0, x_0) - \frac{c_2 \Delta K(x_0)}{36 K(x_0)} > 0$, *where* c_1 *and* c_2 *are the constants defined in Theorem 1.4,*
(iii) $n \geq 7$ *and* $-\Delta K(x_0) > 0$.
Then the problem

$$(Q_\varepsilon) \quad \begin{cases} \Delta^2 u = K u^{p+\varepsilon}, \ u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution of the form (1.8) *satisfying* (1.9).

The proof of our results is inspired by the methods of [2], [6], [11] and [21]. The next section will be devoted to some useful estimates needed in the proofs of our results. In section 3 we prove Theorems 1.1, 1.2 and 1.5, while Theorems 1.3, 1.4 and 1.6 are proved in section 4. Lastly, we give in the appendix some integral estimates which are needed in Section 2.

2 The Technical Framework

First of all, let us introduce the general setting. For $\varepsilon > 0$, we define on $\mathcal{H} \setminus \{0\}$ the functional

$$J_\varepsilon(u) = \frac{\int_\Omega |\Delta u|^2}{\left(\int_\Omega K(x)|u|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}}}. \tag{2.1}$$

If u is a critical point of J_ε, u satisfies on Ω the equation

$$\Delta^2 u = l_\varepsilon(u) K(x) |u|^{p-1-\varepsilon} u \tag{2.2}$$

with

$$l_\varepsilon(u) = \frac{\int_\Omega |\Delta u|^2}{\int_\Omega K(x)|u|^{p+1-\varepsilon}}. \tag{2.3}$$

Conversely, we see that any solution of (2.2) is a critical point of J_ε.
Note that if u is a positive critical point of J_ε, then $(l_\varepsilon(u))^{\frac{1}{p-1-\varepsilon}} u$ is a solution of (P_ε). This will allow us to look for solutions of (P_ε) as critical points of J_ε.
Now let

$$\mathcal{M}_\varepsilon = \{(x, \lambda, v) \in \Omega \times \mathbb{R}_+^* \times \mathcal{H}(\Omega) / v \in E_{x,\lambda}, \ ||v|| \leq \nu_0\},$$

where ν_0 is a small positive constant.
Let us define the functional

$$\psi_\varepsilon : \mathcal{M}_\varepsilon \to \mathbb{R}, \ \psi_\varepsilon(x, \lambda, v) = J_\varepsilon(P\delta_{x,\lambda} + v). \tag{2.4}$$

Notice that (x, λ, v) is a critical point of ψ_ε if and only if $u = P\delta_{x,\lambda} + v$ is a critical point of J_ε. So this fact allows us to look for critical points of J_ε by successive optimizations with respect to the different parameters on $\mathcal{M}_\varepsilon$.
On the other hand, $(x, \lambda, v) \in \mathcal{M}_\varepsilon$ is a critical point of ψ_ε on $\mathcal{M}_\varepsilon$ if and only if there are A, B, $C_j \in \mathbb{R}$, $1 \le j \le n$, such that

$$(E_{x_i}): \quad \frac{\partial \psi_\varepsilon}{\partial x_i} = B\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda\partial x_i}, v\right) + \sum_{j=1}^n C_j\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial x_j \partial x_i}, v\right), \; i = 1, ..., n$$

$$(E_\lambda): \quad \frac{\partial \psi_\varepsilon}{\partial \lambda} = B\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}, v\right) + \sum_{j=1}^n C_j\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial x_j \partial \lambda}, v\right),$$

$$(E_v): \quad \frac{\partial \psi_\varepsilon}{\partial v} = AP\delta_{x,\lambda} + B\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^n C_j \frac{\partial P\delta_{x,\lambda}}{\partial x_j}.$$

As usual in these types of problems, we first deal with the v-part of u. Namely, we prove the following.

Proposition 2.1 *There exist $\varepsilon_1 > 0$, $\nu_0 > 0$, and a smooth map which to any $(\varepsilon, x, \lambda) \in (0, \varepsilon_1) \times \Omega \times \mathbb{R}_+^*$ with $\lambda d(x, \partial\Omega) > \nu_0^{-1}$, and $\varepsilon \log \lambda < \nu_0$, associates $v_\varepsilon = v_{\varepsilon,x,\lambda} \in E_{x,\lambda}$, $||v_\varepsilon|| < \nu_0$ such that (E_v) is satisfied for some $(A, B, C_1, ..., C_n)_{\varepsilon,x,\lambda} \in \mathbb{R}^{n+2}$. Such a v_ε is unique, minimizes $\psi_\varepsilon(x, \lambda, v)$ with respect to v in $\{v \in E_{x,\lambda}/||v|| < \nu_0\}$, and we have the following estimate*

$$||v_\varepsilon|| \; = O\left(\sum_{j=1}^k \frac{|D^j K(x)|}{\lambda^j} + \frac{1}{\lambda^{k+1}} + \varepsilon + \frac{1}{(\lambda d)^{\frac{n-4}{2}+\theta}}\right),$$

where $\theta > 0$, k is the biggest positive integer satisfying $k \le \frac{n-4}{2}$, and where $d = d(x, \partial\Omega)$.

Proof. As in [2] (see also [3] and [21]) we write

$$\begin{aligned}\psi_\varepsilon(x, \lambda, v) &= J_\varepsilon(P\delta_{x,\lambda} + v) \\ &= \frac{||P\delta_{x,\lambda} + v||^2}{\left(\int_\Omega K(y)|P\delta_{x,\lambda} + v|^{p+1-\varepsilon}\right)^{2/(p+1-\varepsilon)}} \\ &= \psi_\varepsilon(x, \lambda, 0) - (f_\varepsilon, v) + \frac{1}{2}Q_\varepsilon(v, v) + O\left(||v||^{\min(3,p+1-\varepsilon)}\right),\end{aligned} \tag{2.5}$$

where

$$(f_\varepsilon, v) = 2J_\varepsilon(P\delta_{x,\lambda})\frac{\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}v}{\int_\Omega K(y)P\delta_{x,\lambda}^{p+1-\varepsilon}},$$

and

$$\begin{aligned}Q_\varepsilon(v, v) = 2J_\varepsilon(P\delta_{x,\lambda}) \Bigg[&\frac{||v||^2}{||P\delta_{x,\lambda}||^2} - (p - \varepsilon)\frac{\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}v^2}{\int_\Omega K(y)P\delta_{x,\lambda}^{p+1-\varepsilon}} \\ &+(p + 3 - \varepsilon)\left(\frac{\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}v}{\int_\Omega K(y)P\delta_{x,\lambda}^{p+1-\varepsilon}}\right)^2\Bigg].\end{aligned}$$

It follows from Proposition 2.1 [9], and Lemmas 5.1 and 5.2 that

$$\int_\Omega K(y) P\delta_{x,\lambda}^{p+1-\varepsilon} = K(x) S_n + O\left(\frac{1}{\lambda^2} + \varepsilon \log \lambda + \frac{1}{(\lambda d)^{n-4}}\right), \tag{2.6}$$

$$\int_\Omega K(y) P\delta_{x,\lambda}^{p-\varepsilon} v = O\left(\varepsilon + \sum_{j=1}^{k} \frac{|D^j K(x)|}{\lambda^j} + \frac{1}{\lambda^{k+1}} + \frac{1}{(\lambda d)^{\frac{n-4}{2}+\theta}}\right) ||v||, \tag{2.7}$$

where k denotes the biggest positive integer satisfying $k \leq n-4/2$ and $\theta > 0$.
Now, we observe that

$$\begin{aligned} \int_\Omega K(y) P\delta_{x,\lambda}^{p-1-\varepsilon} v^2 &= \int_\Omega K(y) \delta_{x,\lambda}^{p-1-\varepsilon} v^2 + o(||v||^2) \\ &= \int_\Omega K(y) \delta_{x,\lambda}^{p-1} v^2 + o(||v||^2) \\ &= K(x) \int_\Omega \delta_{x,\lambda}^{p-1} v^2 + o(||v||^2). \end{aligned} \tag{2.8}$$

One can check that (see [9])

$$||P\delta_{x,\lambda}||^2 = S_n + O((\lambda d)^{4-n}). \tag{2.9}$$

Combining (2.6),..., (2.9), we obtain

$$Q_\varepsilon(v,v) = \frac{2J_\varepsilon(P\delta_{x,\lambda})}{S_n}\left[||v||^2 - p\int_\Omega \delta_{x,\lambda}^{p-1} v^2 + o(||v||^2)\right]. \tag{2.10}$$

According to [5], there exists some positive constant independent of ε, for ε small enough, such that

$$||v||^2 - p\int_\Omega \delta_{x,\lambda}^{p-1} v^2 \geq c||v||^2, \quad \forall v \in E_{x,\lambda}. \tag{2.11}$$

It follows from Lemma 5.2 that

$$(f_\varepsilon, v) = O\left(\sum_{j=1}^{k} \frac{|D^j K(x)|}{\lambda^j} + \frac{1}{\lambda^{k+1}} + \varepsilon + \frac{1}{(\lambda d)^{\frac{n-4}{2}+\theta}}\right). \tag{2.12}$$

It is easy to see that Proposition 2.1 follows from (2.5),..., (2.12). □

Next, we prove a useful expansion of the functional J_ε associated to (P_ε), and its gradient in a neighborhood of potential concentration sets.

Proposition 2.2 *Suppose that $\lambda d(x,\partial\Omega)\to+\infty$ and $\varepsilon\log\lambda\to 0$ as $\varepsilon\to 0$. Then we have the following expansion*

$$
\begin{aligned}
J_\varepsilon(P\delta_{x,\lambda}) = \frac{S_n^{\frac{p-1-\varepsilon}{p+1-\varepsilon}}}{K(x)^{\frac{2}{p+1-\varepsilon}}}\Bigg[1 &- \frac{(n-4)c_2\Delta K(x)}{2n^2 S_n K(x)\lambda^2} \\
&+ \frac{n-4}{n}\varepsilon\left(\log\lambda^{\frac{n-4}{2}} + \frac{c_3}{S_n}\right) + \frac{c_1 H(x,x)}{S_n\lambda^{n-4}} \\
&+ O\left(\frac{\varepsilon\log\lambda}{\lambda^2} + \frac{1}{\lambda^{n-3}} + \frac{1}{(\lambda d)^{n-2}} + \sum_{j=3}^{n-4}\frac{|D^j K(x)|}{\lambda^j}\right) \\
&+ O\left(\frac{\varepsilon\log\lambda}{(\lambda d)^{n-4}} + \varepsilon^2\log^2\lambda + (\ \text{if } n<8)\frac{1}{(\lambda d)^{2(n-4)}}\right)\Bigg],
\end{aligned}
$$

where S_n, c_1, c_2 and c_3 are defined in Lemma 5.1.

Proof. According to [9], we have

$$
||P\delta_{x,\lambda}||^2 = S_n - c_1\frac{H(x,x)}{\lambda^{n-4}} + O\left(\frac{1}{(\lambda d)^{n-2}}\right). \tag{2.13}
$$

We also have

$$
\begin{aligned}
\int_\Omega K(y)P\delta_{x,\lambda}^{p+1-\varepsilon} &= \int_\Omega K(y)\left(\delta_{x,\lambda}-\varphi_{x,\lambda}\right)^{p+1-\varepsilon} \\
&= \int_\Omega K(y)\delta_{x,\lambda}^{p+1-\varepsilon} - (p+1-\varepsilon)\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\varphi_{x,\lambda} \\
&\quad + O\left(\int_{B(x,d)}\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}^2 + \frac{1}{(\lambda d)^{n-1}}\right).
\end{aligned} \tag{2.14}
$$

We now observe that, for $n\geq 8$, we have $\frac{n}{n-4}\leq 2$ and thus in this case we have

$$
\begin{aligned}
\int_{B(x,d)}\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}^2 &\leq \int_{B(x,d)}\delta_{x,\lambda}^{\frac{n}{n-4}}\varphi_{x,\lambda}^{\frac{n}{n-4}-\varepsilon} \\
&\leq ||\varphi_{x,\lambda}||_\infty^{\frac{n}{n-4}-\varepsilon}\int_{B(x,d)}\delta_{x,\lambda}^{\frac{n}{n-4}} \\
&= O\left(\frac{(\lambda d^2)^{\varepsilon\frac{n-4}{2}}\log(\lambda d)}{(\lambda d)^n}\right) = O\left(\frac{1}{(\lambda d)^{n-1}}\right),
\end{aligned} \tag{2.15}
$$

and, for $n<8$, we have

$$
\int_{B(x,d)}\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}^2 \leq \frac{1}{(\lambda d^2)^{(n-4)}}\int_{B(x,d)}\delta_{x,\lambda}^{p-1-\varepsilon} = O\left(\frac{1}{(\lambda d)^{2(n-4)}}\right). \tag{2.16}
$$

Thus using Lemma 5.1 and (2.15) , (2.16), we obtain

$$\int_\Omega K(y)P\delta_{x,\lambda}^{p+1-\varepsilon} = K(x)S_n + \frac{c_2\Delta K(x)}{2n\lambda^2} - \varepsilon K(x)S_n\left(\log\lambda^{\frac{n-4}{2}} + \frac{c_3}{S_n}\right) - \frac{c_1 2nK(x)H(x,x)}{(n-4)\lambda^{n-4}}$$
$$+ O\left(\frac{\varepsilon\log\lambda}{(\lambda d)^{n-4}} + \frac{1}{(\lambda d)^{n-2}} + \sum_{j=3}^{n-4}\frac{|D^jK(x)|}{\lambda^j} + \frac{1}{\lambda^{n-3}}\right)$$
$$+ O\left(\frac{\varepsilon\log\lambda}{\lambda^2} + (\varepsilon\log\lambda)^2 + (if n<8)\frac{1}{(\lambda d)^{2(n-4)}}\right). \tag{2.17}$$

(2.13), (2.17) obviously show that Proposition 2.2 holds. □

The following lemma gives the basic property of the functional l_ε defined in (2.3).

Lemma 2.3 *Assume that $x \in \Omega$ such that $d = d(x, \partial\Omega) \geq d_0 > 0$, and let v_ε be the function obtained in Proposition 2.1. Then the functional l_ε has the following expansion :*

$$l_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon) = \frac{1}{K(x)}\left[1 + O\left(\frac{1}{\lambda^{n-4}} + \varepsilon\log\lambda + \sum_{j=2}^{n-4}\frac{|D^jK(x)|}{\lambda^j} + \sum_{j=1}^{k}\frac{|D^jK(x)|^2}{\lambda^{2j}} + \frac{1}{\lambda^{2k+2}}\right)\right],$$

where k is the biggest positive integer satisfying $k \leq \frac{n-4}{2}$.

Proof. We have

$$||P\delta_{x,\lambda} + v_\varepsilon||^2 = ||P\delta_{x,\lambda}||^2 + ||v_\varepsilon||^2. \tag{2.18}$$

We also have

$$\int_\Omega K(y)|P\delta_{x,\lambda} + v_\varepsilon|^{p+1-\varepsilon} = \int_\Omega K(y)P\delta_{x,\lambda}^{p+1-\varepsilon} + (p+1-\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}v_\varepsilon + O(||v_\varepsilon||^2). \tag{2.19}$$

Thus, using (2.13), (2.17), (2.18), (2.19) Lemma 5.2 and Proposition 2.1, we easily derive our lemma. □

Lemma 2.4 *Assume that $x \in \Omega$ such that $d = d(x, \partial\Omega) \geq d_0 > 0$, and let v_ε be the function obtained in Proposition 2.1. Then the following expansion holds.*

$$\left(\nabla J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon), \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right) = \frac{1}{(S_nK(x))^{\frac{2}{p+1-\varepsilon}}}\left[\frac{c_2(n-4)\Delta K(x)}{n^2K(x)\lambda^3} - \frac{c_1(n-4)H(x,x)}{\lambda^{n-3}}\right.$$
$$+ \frac{(n-4)^2S_n\varepsilon}{2n\lambda} + O\left(\frac{\varepsilon\log\lambda}{\lambda^3} + \frac{1}{\lambda^{n-2}} + \frac{\varepsilon^2\log\lambda}{\lambda} + \frac{\varepsilon\log\lambda}{\lambda^{n-3}}\right)$$
$$\left. + O\left(\sum_3^{n-4}\frac{|D^jK(x)|}{\lambda^{j+1}} + \sum_1^k\frac{|D^jK(x)|^2}{\lambda^{2j+1}} + \frac{1}{\lambda^{2k+3}} + (if n<8)\frac{1}{\lambda^{2n-7}}\right)\right],$$

where k is the biggest positive integer satisfying $k \leq \frac{n-4}{2}$.

Proof. We have

$$\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon),\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)=\frac{2}{\left(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}}}\Bigg[\left(P\delta_{x,\lambda},\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)$$
$$-l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\Bigg]. \qquad (2.20)$$

According to [9], we have

$$\left(P\delta_{x,\lambda},\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)=\frac{c_1(n-4)H(x,x)}{2\lambda^{n-3}}+O\left(\frac{1}{\lambda^{n-1}}\right). \qquad (2.21)$$

On the other hand it follows from Lemma 5.2 and Proposition 2.1 that

$$\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}=\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}$$
$$+(p-\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}v_\varepsilon\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+O\left(\frac{||v_\varepsilon||^2}{\lambda}\right)$$
$$=\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+O\left(\sum_1^k\frac{|D^jK(x)|^2}{\lambda^{2j+1}}+\frac{1}{\lambda^{2k+3}}+\frac{1}{\lambda^{n-3+2\theta}}+\frac{\varepsilon^2}{\lambda}\right). \qquad (2.22)$$

We are now going to estimate the integral in the right-hand side of (2.22). To this aim, we write

$$\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}=\int_\Omega K(y)(\delta_{x,\lambda}-\varphi_{x,\lambda})^{p-\varepsilon}\frac{\partial(\delta_{x,\lambda}-\varphi_{x,\lambda})}{\partial\lambda}$$
$$=\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial\delta_{x,\lambda}}{\partial\lambda}-\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}$$
$$-(p-\varepsilon)\int_\Omega K(y)\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}\frac{\partial\delta_{x,\lambda}}{\partial\lambda}+O\left(\int_\Omega\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}|\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}|\right)$$
$$+O\left(\int_\Omega\delta_{x,\lambda}^{p-1-\varepsilon}\frac{\varphi_{x,\lambda}^2}{\lambda}+\frac{1}{\lambda^{n-1}}\right). \qquad (2.23)$$

As in (2.15) and(2.16) we derive that

$$O\left(\int_\Omega\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}|\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}|\right)=O\left(\int_\Omega\delta_{x,\lambda}^{p-1-\varepsilon}\frac{\varphi_{x,\lambda}^2}{\lambda}\right)=O\left(\frac{1}{\lambda^n}+(ifn<8)\,\frac{1}{\lambda^{2n-7}}\right). \qquad (2.24)$$

Lemma 2.4 follows from (2.20),..., (2.24) and Lemmas 2.3, 5.1. □

Lemma 2.5 *Suppose that K satisfies the assumptions of Theorem 1.3 and*

$$|x-x_0|\le\varepsilon^{1/L},\quad \lambda\in[c\varepsilon^{-1/(n-4)},c'\varepsilon^{-1/(n-4)}]. \qquad (2.25)$$

Then

$$||v_\varepsilon||=O\left(\varepsilon^{(1+\sigma)/2}\right),$$

where σ is a positive constant and where v_ε is defined in Proposition 2.1.

Proof. It view of Proposition 2.1, we only need to check

$$\frac{|D^jK(x)|}{\lambda^j} = O(\varepsilon^{1+\sigma}). \tag{2.26}$$

But, by assumptions imposed on K, we see that if $\sigma > 0$ is small enough, then

$$\frac{|D^jK(x)|}{\lambda^j} \leq C\frac{|x-x_0|^{L-j}}{\lambda^j} \leq C\varepsilon^{\frac{L-j}{L}}\varepsilon^{\frac{j}{n-4}} = O\left(\varepsilon^{1+\sigma}\right).$$

□

Lemma 2.6 *Suppose that K satisfies the assumptions of Theorem 1.3 and* (2.25) *holds. Then we have the following estimates:*

$$\begin{array}{lrcl} 1. & (\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), P\delta_{x,\lambda}) & = & O\left(\varepsilon^{1-\sigma}\right) \\ 2. & \left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right) & = & O\left(\varepsilon^{1+\frac{1}{n-4}}\right) \\ 3. & \left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), \frac{\partial P\delta_{x,\lambda}}{\partial x_j}\right) & = & O\left(\varepsilon^{1+\sigma-\frac{1}{n-4}}\right), \end{array}$$

where v_ε is defined in Proposition 2.1.

Proof. Lemma 2.4 and (2.26) give Claim 2. To prove Claim 1, we write

$$\begin{aligned}(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), P\delta_{x,\lambda}) = & \frac{2}{\left(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}}}\Big[(P\delta_{x,\lambda}, P\delta_{x,\lambda}) \\ & -l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}P\delta_{x,\lambda}\Big]\end{aligned}$$

and thus, using Lemmas 5.2, 2.3, Proposition 2.1, (2.13), (2.17) and (2.26) we easily derive Claim 1.
As in (2.21), (2.23) (see also [9])we have

$$\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right) = \frac{\partial H(x,x)}{\partial x_i}\frac{c_1}{2\lambda^{n-4}} + O\left(\frac{1}{\lambda^{n-2}}\right). \tag{2.27}$$

$$\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial x_i} = O\left(\sum_1^{n-4}\frac{|D^jK(x)|}{\lambda^{j-1}} + \frac{1}{\lambda^{n-4}} + (if n<8)\frac{1}{\lambda^{2n-9}}\right). \tag{2.28}$$

Then Claim 3 follows. □

Next, our goal is to estimate $||\partial v_\varepsilon/\partial\lambda||$, where v_ε is defined in Proposition 2.1. To this aim, we follow [11], namely, we write the following decomposition

$$\frac{\partial v_\varepsilon}{\partial\lambda} = w + \alpha P\delta_{x,\lambda} + \beta\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^n \gamma_j\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \tag{2.29}$$

where α, β and γ_j are chosen in such a way that $w \in E_{x,\lambda}$.

Lemma 2.7 *Let α, β and γ_j be coefficients in* (2.29) *and assume that* (2.25) *holds. Then we have the following estimates*

$$\alpha = O\left(\frac{||v_\varepsilon||}{\lambda}\right), \quad \beta = O(||v_\varepsilon||), \quad \gamma_j = O\left(\frac{||v_\varepsilon||}{\lambda^2}\right).$$

Proof. Taking the scalar product of (2.29) with $P\delta_{x,\lambda}$, $\partial P\delta_{x,\lambda}/\partial\lambda$ and $\partial P\delta_{x,\lambda}/\partial x_i$ for $i = 1, ..., n$, we obtain

$$\alpha||P\delta_{x,\lambda}||^2 + \beta\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, P\delta_{x,\lambda}\right) + \sum_{j=1}^{n}\gamma_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, P\delta_{x,\lambda}\right) = 0,$$

$$\begin{aligned}\alpha\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right) + \beta\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|\right|^2 + \sum_{j=1}^{n}\gamma_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)\\ = -\left(v_\varepsilon, \frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right) = O\left(\frac{||v_\varepsilon||}{\lambda^2}\right),\end{aligned}$$

$$\begin{aligned}\alpha\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right) + \beta\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right) + \sum_{j=1}^{n}\gamma_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)\\ = -\left(v_\varepsilon, \frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda\partial x_i}\right) = O\left(||v_\varepsilon||\right).\end{aligned}$$

Thus, we derive that

$$\begin{aligned}\frac{\alpha}{\lambda}(S_n + O(\varepsilon)) + \frac{\beta}{\lambda^2}O(\varepsilon) + \sum_{1}^{n}\gamma_j O(\varepsilon) = 0\\ \frac{\alpha}{\lambda}O(\varepsilon) + \frac{\beta}{\lambda^2}(c'_n + O(\varepsilon)) + \sum_{1}^{n}\gamma_j O(\varepsilon) = O\left(\frac{||v_\varepsilon||}{\lambda^2}\right)\\ \frac{\alpha}{\lambda}O(\varepsilon) + \frac{\beta}{\lambda^2}O(\varepsilon) + \sum_{j\neq i}\gamma_j O(\varepsilon) + \gamma_i(c''_n + O(\varepsilon)) = O\left(\frac{||v_\varepsilon||}{\lambda^2}\right).\end{aligned}$$

Solving the above system we get the desired estimates. $\square$

Now, for a fixed $w_0 \in E_{x_0,\lambda_0}$, we denote $\pi(x,\lambda)$ the orthogonal projection of w_0 onto $E_{x,\lambda}$. We then have

$$w_0 = \pi(x,\lambda) + a(x,\lambda)P\delta_{x,\lambda} + b(x,\lambda)\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^{n} g_j(x,\lambda)\frac{\partial P\delta_{x,\lambda}}{\partial x_j}. \tag{2.30}$$

Lemma 2.8 *The map $\pi(.,.)$ is C^1 with respect to x and λ, and*

$$a(x_0,\lambda_0)=0, \qquad \frac{\partial a(x_0,\lambda_0)}{\partial \lambda}=O\left(\frac{||w_0||}{\lambda}\right),$$
$$b(x_0,\lambda_0)=0, \qquad \frac{\partial b(x_0,\lambda_0)}{\partial \lambda}=O\left(||w_0||\right),$$
$$g_j(x_0,\lambda_0)=0, \qquad \frac{\partial g_j(x_0,\lambda_0)}{\partial \lambda}=O\left(\frac{||w_0||}{\lambda^2}\right).$$

Proof. First of all, we easily deduce from the fact that $w_0\in E_{x_0,\lambda_0}$ the following:

$$a(x_0,\lambda_0)=b(x_0,\lambda_0)=g_j(x_0,\lambda_0)=0.$$

Secondly, it is clear to see that $a(x,\lambda)$, $b(x,\lambda)$ and $g_j(x,\lambda)$ satisfy

$$a||P\delta_{x,\lambda}||^2+b\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},P\delta_{x,\lambda}\right)+\sum_{j=1}^n g_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j},P\delta_{x,\lambda}\right)=\left(w_0,P\delta_{x,\lambda}\right), \tag{2.31}$$

$$a\left(P\delta_{x,\lambda},\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)+b\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|\right|^2+\sum_{j=1}^n g_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j},\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)=\left(w_0,\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right), \tag{2.32}$$

$$a\left(P\delta_{x,\lambda},\frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)+b\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},\frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)+\sum_{j=1}^n g_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j},\frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)=\left(w_0,\frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right). \tag{2.33}$$

Solving the above system we easily see that $a(x,\lambda)$, $b(x,\lambda)$ and $g_j(x,\lambda)$ are C^1 with respect to x and λ. Differentiating (2.31), (2.32) and (2.33) with respect to λ, we obtain

$$\frac{\partial a(x_0,\lambda_0)}{\partial\lambda}||P\delta_{x_0,\lambda_0}||^2+\frac{\partial b(x_0,\lambda_0)}{\partial\lambda}\left(\frac{\partial P\delta_{x_0,\lambda_0}}{\partial\lambda},P\delta_{x_0,\lambda_0}\right)+\sum_{j=1}^n\frac{\partial g_j(x_0,\lambda_0)}{\partial\lambda}$$
$$\times\left(\frac{\partial P\delta_{x_0,\lambda_0}}{\partial x_j},P\delta_{x_0,\lambda_0}\right)=\left(w_0,\frac{\partial P\delta_{x_0,\lambda_0}}{\partial\lambda}\right)=0,$$

$$\frac{\partial a(x_0,\lambda_0)}{\partial\lambda}\left(P\delta_{x_0,\lambda_0},\frac{\partial P\delta_{x_0,\lambda_0}}{\partial\lambda}\right)+\frac{\partial b(x_0,\lambda_0)}{\partial\lambda}\left|\left|\frac{\partial P\delta_{x_0,\lambda_0}}{\partial\lambda}\right|\right|^2+\sum_{j=1}^n\frac{\partial g_j(x_0,\lambda_0)}{\partial\lambda}$$
$$\times\left(\frac{\partial P\delta_{x_0,\lambda_0}}{\partial x_j},\frac{\partial P\delta_{x_0,\lambda_0}}{\partial\lambda}\right)=\left(w_0,\frac{\partial^2 P\delta_{x_0,\lambda_0}}{\partial\lambda^2}\right)=O\left(\frac{||w_0||}{\lambda^2}\right),$$

$$\frac{\partial a(x_0,\lambda_0)}{\partial\lambda}\left(P\delta_{x_0,\lambda_0},\frac{\partial P\delta_{x_0,\lambda_0}}{\partial x_i}\right)+\frac{\partial b(x_0,\lambda_0)}{\partial\lambda}\left(\frac{\partial P\delta_{x_0,\lambda_0}}{\partial\lambda},\frac{\partial P\delta_{x_0,\lambda_0}}{\partial x_i}\right)$$
$$+\sum_{j=1}^n\frac{\partial g_j(x_0,\lambda_0)}{\partial\lambda}\left(\frac{\partial P\delta_{x_0,\lambda_0}}{\partial x_j},\frac{\partial P\delta_{x_0,\lambda_0}}{\partial x_i}\right)=\left(w_0,\frac{\partial^2 P\delta_{x_0,\lambda_0}}{\partial\lambda\partial x_i}\right)=O(||w_0||).$$

Thus, as in the proof of Lemma 2.7, we derive the desired result. □

Proposition 2.9 *Assume that* (2.25) *holds. Then, we have the following estimate*

$$\left|\left|\frac{\partial v_\varepsilon}{\partial \lambda}\right|\right| = O\left(\varepsilon^{\frac{1+\sigma}{2}+\frac{1}{n-4}}\right),$$

where v_ε *is defined in Proposition 2.1.*

Proof. In view of (2.29) and Lemma 2.7, we only need to estimate $||w||$. Let $\pi(x',\lambda')$ be the orthogonal projection of $w \in E_{x,\lambda}$ onto $E_{x',\lambda'}$. Thus we have

$$\left(\nabla J_\varepsilon\left(P\delta_{x',\lambda'} + v_\varepsilon(x',\lambda')\right), \pi(x',\lambda')\right) = 0. \tag{2.34}$$

Differentiating (2.34) with respect to λ' and letting $(x',\lambda') = (x,\lambda)$, we obtain

$$D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda} + \frac{\partial v_\varepsilon}{\partial \lambda}, w\right) + \left(\nabla J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon), \frac{\partial \pi(x,\lambda)}{\partial \lambda}\right) = 0. \tag{2.35}$$

It follows from Lemmas 2.6 and 2.8 that

$$\begin{aligned}
\left(\nabla J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon), \frac{\partial \pi(x,\lambda)}{\partial \lambda}\right) &= \frac{\partial a}{\partial \lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon), P\delta_{x,\lambda}\right) \\
&+ \frac{\partial b}{\partial \lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon), \frac{\partial P\delta_{x,\lambda}}{\partial \lambda}\right) + \sum_1^n \frac{\partial g_j}{\partial \lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon), \frac{\partial P\delta_{x,\lambda}}{\partial x_j}\right) \\
&= O\left(||w||\left(\frac{\varepsilon^{1-\sigma}}{\lambda} + \varepsilon^{1-\sigma+\frac{1}{n-4}} + \frac{\varepsilon^{1-\sigma-\frac{1}{n-4}}}{\lambda^2}\right)\right) = O\left(||w||\varepsilon^{1-\sigma+\frac{1}{n-4}}\right).
\end{aligned} \tag{2.36}$$

Combining (2.34) and (2.35) and taking Lemmas 2.6 and 2.7 into account we obtain

$$\begin{aligned}
D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)(w,w) &= -D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)\Bigg(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda} + \alpha P\delta_{x,\lambda} + \beta\frac{\partial P\delta_{x,\lambda}}{\partial \lambda} \\
&+ \sum_1^n \gamma_j \frac{\partial P\delta_{x,\lambda}}{\partial x_j}, w\Bigg) + O\left(||w||\varepsilon^{1-\sigma+\frac{1}{n-4}}\right) \\
&= -D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda}, w\right) + O\left(\frac{||w||||v_\varepsilon||}{\lambda}\right) + O\left(||w||\varepsilon^{1-\sigma+\frac{1}{n-4}}\right) \\
&= -D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda}, w\right) + O\left(||w||\varepsilon^{\frac{1+\sigma}{2}+\frac{1}{n-4}}\right).
\end{aligned} \tag{2.37}$$

We now claim that

$$D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)(w,w) \geq \rho||w||^2, \tag{2.38}$$

for some positive constant ρ and

$$D^2 J_\varepsilon(P\delta_{x,\lambda} + v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda}, w\right) = O\left(||w||\varepsilon^{\frac{1+\sigma}{2}+\frac{1}{n-4}}\right). \tag{2.39}$$

Then obviously (2.37), (2.38) and (2.39) imply that $||w|| = O\left(\varepsilon^{\frac{1+\sigma}{2}+\frac{1}{n-4}}\right)$, and Proposition 2.9 follows.

It remains to prove (2.38) and (2.39). To this aim, we write

$$
\begin{aligned}
D^2 J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)(\varphi,\psi) &= \frac{2(\varphi,\psi)}{\left(\int_\Omega K(y)|u|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}}} - \frac{4(P\delta_{x,\lambda}+v_\varepsilon,\varphi)}{\left(\int_\Omega K(y)|u|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}+1}} \\
&\times \int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\psi - \frac{4(P\delta_{x,\lambda}+v_\varepsilon,\psi)}{\left(\int_\Omega K(y)|u|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}+1}} \int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\varphi \\
&+ 2(p+3-\varepsilon)||P\delta_{x,\lambda}+v_\varepsilon||^2 \frac{\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\varphi \int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\psi}{\left(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}+2}} \\
&- \frac{2(p-\varepsilon)||P\delta_{x,\lambda}+v_\varepsilon||^2}{\left(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}+1}} \int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\varphi\psi. \qquad (2.40)
\end{aligned}
$$

Verification of (2.38). First, we notice that

$$(P\delta_{x,\lambda}+v_\varepsilon, w) = (v_\varepsilon, w) = O\left(\varepsilon^{(1+\sigma)/2}||w||\right), \qquad (2.41)$$

where we have used Lemma 2.5. By Lemma 5.2, we see

$$\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}w = \int_\Omega K(y)|P\delta_{x,\lambda}|^{p-\varepsilon}w + O\left(||v_\varepsilon||||w||\right) = O\left(\varepsilon^{\frac{1+\sigma}{2}}||w||\right). \qquad (2.42)$$

Thus (2.38) follows from (2.40),...,(2.42) and Proposition 3.4 in [5].

Verification of (2.39). We have

$$\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, w\right) = 0, \qquad (2.43)$$

$$\left(P\delta_{x,\lambda}+v_\varepsilon, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right) = \left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right) = O\left(\frac{1}{\lambda^{n-3}}\right) = O\left(\varepsilon^{1+\frac{1}{n-4}}\right). \qquad (2.44)$$

Also, it follows from(2.22) that

$$\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} = O\left(\varepsilon^{1+\frac{1}{n-4}}\right), \qquad (2.45)$$

and by Lemma 5.2 as in (2.42), we have

$$
\begin{aligned}
&\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}w = \int_\Omega K(y)|P\delta_{x,\lambda}|^{p-1-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}w + O\left(\frac{||v_\varepsilon||||w||}{\lambda}\right) \\
&= O\left(||w||\varepsilon^{\frac{1+\sigma}{2}+\frac{1}{n-4}}\right). \qquad (2.46)
\end{aligned}
$$

Combining (2.41),..., (2.46) we obtain (2.39) and this completes the proof of Proposition 2.9. □

Lemma 2.10 *The derivative of the functional J_ε satisfies*

$$(i)\quad \frac{\partial}{\partial\lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon),\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)=\frac{1}{(K(x)S_n)^{\frac{2}{p+1-\varepsilon}}}\Bigg[-\frac{(n-4)^2S_n\varepsilon}{2n\lambda^2}$$
$$+\frac{c_1(n-4)(n-3)H(x,x)}{\lambda^{n-2}}+O\left(\varepsilon^{1+\sigma+2/(n-4)}\right)\Bigg],$$

$$(ii)\quad \frac{\partial}{\partial\lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon),\frac{\partial P\delta_{x,\lambda}}{\partial x_j}\right)=O\left(\varepsilon^{1-\sigma}\right),$$

$$(iii)\quad \frac{\partial}{\partial\lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon),P\delta_{x,\lambda}\right)=O\left(\varepsilon^{1+1/(n-4)}\right).$$

Proof. By easy computations we have

$$\frac{\partial}{\partial\lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon),\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)=D^2J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda},\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)$$
$$+\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon),\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right).$$

First, we estimate $D^2J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},\frac{\partial v_\varepsilon}{\partial\lambda}\right)$. Using Proposition 2.9, we obtain

$$\left(P\delta_{x,\lambda}+v_\varepsilon,\frac{\partial v_\varepsilon}{\partial\lambda}\right)=\left(v_\varepsilon,\frac{\partial v_\varepsilon}{\partial\lambda}\right)=O\left(\varepsilon^{1+\sigma+\frac{1}{n-4}}\right). \tag{2.47}$$

As in the proof of Lemma 2.4, we have

$$\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\frac{\partial v_\varepsilon}{\partial\lambda}=O\left(\varepsilon^{(1+\sigma)/2+1/(n-4)}\right). \tag{2.48}$$

Combining (2.44), (2.45), (2.47), (2.48), we obtain

$$D^2J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},\frac{\partial v_\varepsilon}{\partial\lambda}\right)=\frac{2}{\left(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}}}\Bigg[\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},\frac{\partial v_\varepsilon}{\partial\lambda}\right)$$
$$-(p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\frac{\partial v_\varepsilon}{\partial\lambda}\Bigg]+O\left(\varepsilon^{1+\sigma+\frac{2}{n-4}}\right). \tag{2.49}$$

We now notice that

$$\frac{\partial v_\varepsilon}{\partial\lambda}=w+\alpha P\delta_{x,\lambda}+\beta\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+\sum_{j=1}^n\gamma_j\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \tag{2.50}$$

$$\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},w\right)=0. \tag{2.51}$$

Consequently, it follows from Lemma 5.2 and Proposition 2.9 that

$$\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}w=O\left(\varepsilon^{1+\sigma+2/(n-4)}\right). \tag{2.52}$$

Now, using Lemmas 5.2, and 2.7, we obtain

$$\alpha \int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon} P\delta_{x,\lambda}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} = O\left(\varepsilon^{1+\sigma+2/(n-4)}\right) \tag{2.53}$$

$$\alpha\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, P\delta_{x,\lambda}\right) = O\left(\varepsilon^{1+\sigma+2/(n-4)}\right). \tag{2.54}$$

In the same way, we have

$$\gamma_j \int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon} \frac{\partial P\delta_{x,\lambda}}{\partial x_j}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} = O\left(\varepsilon^{1+\sigma+2/(n-4)}\right) \tag{2.55}$$

$$\gamma_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right) = O\left(\varepsilon^{1+\sigma+2/(n-4)}\right). \tag{2.56}$$

As in (2.22), (2.23) and using Lemma 2.3 we have

$$\begin{aligned}\beta(p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2 &= \beta(p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\\ \times\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2 + O\left(\frac{\beta||v_\varepsilon||}{\lambda^2}\right) &= \beta\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|\right|^2 + O\left(\varepsilon^{1+\sigma+2/(n-4)}\right),\end{aligned}$$

then, by Lemma 2.7, we derive that

$$\beta\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|\right|^2 - \beta(p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2 = O\left(\varepsilon^{1+\sigma+2/(n-4)}\right). \tag{2.57}$$

Combining (2.47),..., (2.57) we obtain

$$D^2J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, \frac{\partial v_\varepsilon}{\partial\lambda}\right) = O\left(\varepsilon^{1+\sigma+2/(n-4)}\right). \tag{2.58}$$

We now write

$$\begin{aligned}&D^2J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right) + \left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), \frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)\\ &= \frac{2}{(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon})^{\frac{2}{p+1-\varepsilon}}}\Bigg\{\Bigg[\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|\right|^2 - (p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\\ &\times\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-1-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2\Bigg] + \left(P\delta_{x,\lambda}+v_\varepsilon, \frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)\\ &- l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p-\varepsilon}\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\Bigg\} + O\left(\varepsilon^{1+\sigma+2/(n-4)}\right).\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\left(\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon}\right)^{\frac{2}{p+1-\varepsilon}}}\Bigg\{\left\|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right\|^2+\left(P\delta_{x,\lambda},\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)\\
&- l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left((p-\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2+\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)\\
&-(p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\Bigg((p-1-\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-2-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2 v_\varepsilon\\
&+\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}v_\varepsilon\Bigg)+\left(v_\varepsilon,\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)\Bigg\}+O\left(\varepsilon^{1+\sigma+2/(n-4)}\right). \qquad (2.59)
\end{aligned}
$$

We now observe that

$$
\left\|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right\|^2=\int_\Omega\Delta^2\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}=p\int_\Omega\delta_{x,\lambda}^{p-1}\frac{\partial\delta_{x,\lambda}}{\partial\lambda}\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, \qquad (2.60)
$$

$$
\begin{aligned}
\left(P\delta_{x,\lambda},\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)&=\int_\Omega\Delta^2\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)P\delta_{x,\lambda}=p(p-1)\int_\Omega\delta_{x,\lambda}^{p-2}\left(\frac{\partial\delta_{x,\lambda}}{\partial\lambda}\right)^2P\delta_{x,\lambda}\\
&\quad+p\int_\Omega\delta_{x,\lambda}^{p-1}\frac{\partial^2\delta_{x,\lambda}}{\partial\lambda^2}P\delta_{x,\lambda}. \qquad (2.61)
\end{aligned}
$$

Thus, using (2.60), (2.61) and Proposition 2.1 of [9], we obtain

$$
\left\|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right\|^2+\left(P\delta_{x,\lambda},\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)=-\frac{c_1(n-3)(n-4)H(x,x)}{2\lambda^{n-2}}+O\left(\frac{1}{\lambda^n}\right). \qquad (2.62)
$$

As in the proof of Lemma 5.1 and using Proposition 2.1 of [9] , we find

$$
\begin{aligned}
&(p-\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2+\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\\
&\qquad=\frac{(n-4)^2S_nK(x)\varepsilon}{4n\lambda^2}-\frac{c_1(n-4)(n-3)K(x)H(x,x)}{\lambda^{n-2}}+O\left(\varepsilon^{1+\sigma+\frac{2}{n-4}}\right). \qquad (2.63)
\end{aligned}
$$

On the other hand, using Lemma 2.3 it is easy to check

$$
\begin{aligned}
&-(p-\varepsilon)(p-1-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-2-\varepsilon}\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|^2v_\varepsilon\\
&\qquad-(p-\varepsilon)l_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}v_\varepsilon+\left(v_\varepsilon,\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)\\
&\qquad=-\int_\Omega\frac{\partial^2}{\partial\lambda^2}\left(\Delta^2P\delta_{x,\lambda}\right)v_\varepsilon+\left(v_\varepsilon,\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2}\right)+O\left(\varepsilon^{1+\sigma+\frac{2}{n-4}}\right)\\
&\qquad=O\left(\varepsilon^{1+\sigma+\frac{2}{n-4}}\right). \qquad (2.64)
\end{aligned}
$$

Combining the above estimates, Claim (i) follows. The proof of Claim (ii) is similar to that of Claim (i) and therefore is omitted. To prove Claim (iii), we write

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), P\delta_{x,\lambda}\right)\\
&\quad = D^2 J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda}, P\delta_{x,\lambda}\right)+\left(\nabla J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon), \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)\\
&\quad = D^2 J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda}, P\delta_{x,\lambda}\right)+O\left(\varepsilon^{1+1/(n-4)}\right), \qquad (2.65)
\end{aligned}$$

where we have used Lemma 2.6.
Now, as in the proof of (2.58), we obtain

$$D^2 J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial v_\varepsilon}{\partial\lambda}, P\delta_{x,\lambda}\right)=O\left(\varepsilon^{1+\sigma+1/(n-4)}\right). \qquad (2.66)$$

Computations similar to that in the proof of Claim (i) show that

$$D^2 J_\varepsilon(P\delta_{x,\lambda}+v_\varepsilon)\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, P\delta_{x,\lambda}\right)=O\left(\varepsilon^{1+1/(n-4)}\right). \qquad (2.67)$$

Hence, Claim (iii) follows from (2.65), (2.66) and (2.67). This completes the proof of Lemma 2.10. $\square$

Lemma 2.11 *Let A, B and C_j be the constants in (E_v), that is,*

$$\frac{\partial\psi_\varepsilon}{\partial v}=AP\delta_{x,\lambda}+B\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}+\sum_{j=1}^n C_j\frac{\partial P\delta_{x,\lambda}}{\partial x_j},$$

where ψ_ε is defined by (2.4). Then we have the following estimates

$$A=O\left(\varepsilon^{1-\sigma}\right), \quad B=O\left(\varepsilon^{1-1/(n-4)}\right), \quad C_j=O\left(\varepsilon^{1-\sigma+1/(n-4)}\right), \qquad (2.68)$$

$$\frac{\partial A}{\partial\lambda}=O\left(\varepsilon^{1-\sigma+1/(n-4)}\right), \quad \frac{\partial B}{\partial\lambda}=O\left(\varepsilon\right), \quad \frac{\partial C_j}{\partial\lambda}=O\left(\varepsilon^{1-\sigma+2/(n-4)}\right). \qquad (2.69)$$

Proof. By Lemma 2.6, we see that A, B and C_j satisfy

$$A||P\delta_{x,\lambda}||^2+B\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, P\delta_{x,\lambda}\right)+\sum_{j=1}^n C_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, P\delta_{x,\lambda}\right)=O\left(\varepsilon^{1-\sigma}\right), \qquad (2.70)$$

$$A\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)+B\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right|\right|^2+\sum_{j=1}^n C_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \frac{\partial P\delta_{x,\lambda}}{\partial\lambda}\right)=O\left(\varepsilon^{1+\frac{1}{n-4}}\right), \qquad (2.71)$$

$$\begin{aligned}
&A\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)+B\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)+\sum_{j=1}^n C_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)\\
&\qquad\qquad = O\left(\varepsilon^{1-\sigma-\frac{1}{n-4}}\right), \quad \text{for} \quad 1\le i\le n. \qquad (2.72)
\end{aligned}$$

Solving (2.70),..., (2.72), we obtain (2.68). Differentiating (2.70),..., (2.72) with respect to λ and using Lemma 2.10, we obtain

$$\frac{\partial A}{\partial \lambda}||P\delta_{x,\lambda}||^2 + \frac{\partial B}{\partial \lambda}\left(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda}, P\delta_{x,\lambda}\right) + \sum_{j=1}^{n}\frac{\partial C_j}{\partial \lambda}\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, P\delta_{x,\lambda}\right) = O\left(\varepsilon^{1-\sigma+\frac{1}{n-4}}\right), \tag{2.73}$$

$$\frac{\partial A}{\partial \lambda}\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial \lambda}\right) + \frac{\partial B}{\partial \lambda}\left|\left|\frac{\partial P\delta_{x,\lambda}}{\partial \lambda}\right|\right|^2 + \sum_{j=1}^{n}\frac{\partial C_j}{\partial \lambda}\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \frac{\partial P\delta_{x,\lambda}}{\partial \lambda}\right) = O\left(\varepsilon^{1+\frac{2}{n-4}}\right), \tag{2.74}$$

$$\frac{\partial A}{\partial \lambda}\left(P\delta_{x,\lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right) + \frac{\partial B}{\partial \lambda}\left(\frac{\partial P\delta_{x,\lambda}}{\partial \lambda}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right) + \sum_{j=1}^{n}\frac{\partial C_j}{\partial \lambda}\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j}, \frac{\partial P\delta_{x,\lambda}}{\partial x_i}\right)$$
$$= O\left(\varepsilon^{1-\sigma}\right), \quad \text{for} \quad 1 \le i \le n. \tag{2.75}$$

Solving (2.73),..., (2.75), we get (2.69). □

3 Proof of Theorems 1.1, 1.2 and 1.5

First, let us introduce some notations. For two constants β and L such that $L > \beta > 0$ we define a set

$$D_\varepsilon = \{x \in \Omega \cap \overline{B_{\varepsilon^\beta}(x_0)} \mid d(x, \partial\Omega) \ge \varepsilon^L\}. \tag{3.1}$$

For constants $0 < C_0 < C_1$ we set

$$\lambda^\varepsilon_{C_i}(x) = C_i\left(\frac{H(x,x)}{\varepsilon}\right)^{1/(n-4)} \qquad i = 0, 1$$

and we define the following set

$$M_\varepsilon = \{(x,\lambda) \mid x \in D_\varepsilon,\ \lambda \in [\lambda^\varepsilon_{C_0}(x), \lambda^\varepsilon_{C_1}(x)]\}. \tag{3.2}$$

Constants β, L and C_i will be determined later. We now consider the following minimization problem

$$\inf\{\psi_\varepsilon(x, \lambda, v_\varepsilon) \mid (x,\lambda) \in M_\varepsilon\}, \tag{3.3}$$

where v_ε is defined in Proposition 2.1. It is obvious that for small fixed $\varepsilon > 0$ problem (3.3) has a minimizer $(x_\varepsilon, \lambda_\varepsilon)$. In order to prove that $(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon)$ is a critical point of ψ_ε, we only need to prove that $(x_\varepsilon, \lambda_\varepsilon)$ is an interior point of M_ε.

Proof of Theorem 1.1 We prove that if $\varepsilon > 0$ is small enough, the minimizer $(x_\varepsilon, \lambda_\varepsilon)$ of (3.3) is an interior point of M_ε. First we show that if C_0 and C_1 are suitably chosen, then

$$\lambda_\varepsilon \in (\lambda^\varepsilon_{C_0}(x_\varepsilon), \lambda^\varepsilon_{C_1}(x_\varepsilon)). \tag{3.4}$$

Using Proposition 2.1 and the fact that $(x_\varepsilon, \lambda_\varepsilon)$ is a minimum point of (3.3), we obtain

$$\psi_\varepsilon(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon) \le \psi_\varepsilon(x_\varepsilon, \lambda, 0) \quad \text{for all} \quad \lambda \in [\lambda^\varepsilon_{C_0}(x_\varepsilon), \lambda^\varepsilon_{C_1}(x_\varepsilon)]. \tag{3.5}$$

As in the proof of Proposition 2.1, we obtain

$$\begin{aligned}\psi_\varepsilon(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon) &= \psi_\varepsilon(x_\varepsilon,\lambda_\varepsilon,0)+O\left(||v_\varepsilon||^2\right)\\ &= \psi_\varepsilon(x_\varepsilon,\lambda_\varepsilon,0)+O\left(\frac{1}{\lambda_\varepsilon^2}+\varepsilon^2+\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4+2\theta}}\right). \qquad (3.6)\end{aligned}$$

It follows from Proposition 2.2 that

$$\begin{aligned}&\frac{c_1H(x_\varepsilon,x_\varepsilon)}{S_n\lambda_\varepsilon^{n-4}}+\frac{n-4}{n}\varepsilon\left(\log\lambda_\varepsilon^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right)+O\left(\frac{1}{\lambda_\varepsilon^2}+\varepsilon^2\log^2\lambda_\varepsilon+\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{n-4+2\theta}}+\frac{\varepsilon\log\lambda_\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^{n-4}}\right)\\ &\qquad\le\frac{c_1H(x_\varepsilon,x_\varepsilon)}{S_n\lambda^{n-4}}+\frac{n-4}{n}\varepsilon\left(\log\lambda^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right)+O\left(\frac{1}{\lambda^2}+\varepsilon^2\log^2\lambda+\frac{1}{(\lambda d_\varepsilon)^{n-4+2\theta}}\right)\\ &\qquad+O\left(\frac{\varepsilon\log\lambda}{(\lambda d_\varepsilon)^{n-4}}\right). \qquad (3.7)\end{aligned}$$

Since $x_\varepsilon\in D_\varepsilon$, we get $\varepsilon^L\le d_\varepsilon\le\varepsilon^\beta$. If we choose β satisfying

$$\beta>\max\{\frac{1}{2}-\frac{1}{n-4},0\}, \qquad (3.8)$$

then there exists a $\gamma>0$, such that

$$\frac{1}{\lambda^2}\le C\left(\frac{\varepsilon}{H(x_\varepsilon,x_\varepsilon)}\right)^{\frac{2}{n-4}}=O\left(\varepsilon^{\frac{2}{n-4}}d_\varepsilon^2\right)=O\left(\varepsilon^{2\beta+\frac{2}{n-4}}\right)=O\left(\varepsilon^{1+\gamma}\right), \qquad (3.9)$$

$$\varepsilon^2\log^2\lambda=O\left(\varepsilon^2\log\left(\frac{1}{\varepsilon^{\frac{1}{n-4}}d_\varepsilon}\right)\right)=O\left(\varepsilon^2\log\left(\frac{1}{\varepsilon^{\frac{1}{n-4}+L}}\right)\right)=O\left(\varepsilon^{1+\gamma}\right), \qquad (3.10)$$

$$\frac{1}{\lambda d_\varepsilon}\le C\left(\frac{\varepsilon}{H(x_\varepsilon,x_\varepsilon)}\right)^{\frac{1}{n-4}}\frac{1}{d_\varepsilon}=O\left(\varepsilon^{1/(n-4)}\right). \qquad (3.11)$$

Consequently, we have

$$\frac{1}{(\lambda d_\varepsilon)^{n-4+2\theta}}=O\left(\varepsilon^{1+\gamma}\right),\quad \frac{\varepsilon\log\lambda}{(\lambda d_\varepsilon)^{n-4}}=O\left(\varepsilon^{1+\gamma}\right). \qquad (3.12)$$

Inserting (3.9),..., (3.12) into (3.7), we obtain

$$\frac{c_1H(x_\varepsilon,x_\varepsilon)}{S_n\lambda_\varepsilon^{n-4}}+\frac{n-4}{n}\varepsilon\log\lambda_\varepsilon^{\frac{n-4}{2}}\le\frac{c_1H(x_\varepsilon,x_\varepsilon)}{S_n\lambda^{n-4}}+\frac{n-4}{n}\varepsilon\log\lambda^{\frac{n-4}{2}}+O\left(\varepsilon^{1+\gamma}\right). \qquad (3.13)$$

Let

$$\lambda_\varepsilon=t_\varepsilon\left(\frac{H(x_\varepsilon,x_\varepsilon)}{\varepsilon}\right)^{1/(n-4)},\quad \lambda=t\left(\frac{H(x_\varepsilon,x_\varepsilon)}{\varepsilon}\right)^{1/(n-4)}. \qquad (3.14)$$

We then have from (3.13)

$$\frac{c_1}{S_nt_\varepsilon^{n-4}}+\frac{n-4}{n}\log t_\varepsilon^{\frac{n-4}{2}}\le\frac{c_1}{S_nt^{n-4}}+\frac{n-4}{n}\log t^{\frac{n-4}{2}}+O\left(\varepsilon^\gamma\right). \qquad (3.15)$$

Since $t \mapsto \frac{c_1}{S_n t^{n-4}} + \frac{n-4}{n} \log t^{\frac{n-4}{2}}$, $t > 0$, attains its global minimum at

$$t^* = \left(\frac{2nc_1}{(n-4)S_n} \right)^{1/(n-4)}, \tag{3.16}$$

we deduce from (3.15) that as $\varepsilon \to 0$, $t_\varepsilon \to t^*$. If we choose

$$C_0 = \frac{1}{2} \left(\frac{2nc_1}{(n-4)S_n} \right)^{1/(n-4)}, \quad C_1 = \frac{3}{2} \left(\frac{2nc_1}{(n-4)S_n} \right)^{1/(n-4)},$$

then, for $\varepsilon > 0$ small, we obtain (3.4). Now, it remains to prove that x_ε is an interior point of D_ε. To this aim, let ν be the inward unit normal of $\partial\Omega$ at x_0. Let $z_\varepsilon = x_0 + \varepsilon\nu$ and fix $\lambda_\varepsilon^* \in (\lambda_{C_0}^\varepsilon(z_\varepsilon), \lambda_{C_1}^\varepsilon(z_\varepsilon))$. Since $d(z_\varepsilon, \partial\Omega) = \varepsilon$, we have

$$\lambda_\varepsilon^* \sim \left(\frac{1}{\varepsilon \varepsilon^{n-4}} \right)^{1/(n-4)} = \varepsilon^{-(n-3)/(n-4)}.$$

We have

$$\psi_\varepsilon(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon) \le \psi_\varepsilon(z_\varepsilon, \lambda_\varepsilon^*, 0). \tag{3.17}$$

In view of Proposition 2.2, we have

$$\begin{aligned} \psi_\varepsilon(z_\varepsilon, \lambda_\varepsilon^*, 0) &= \frac{S_n^{(p-1-\varepsilon)/(p+1-\varepsilon)}}{(K(z_\varepsilon))^{2/(p+1-\varepsilon)}} \left[1 + O\left(\varepsilon \log(1/\varepsilon)\right)\right] \\ &= \frac{S_n^{(p-1-\varepsilon)/(p+1-\varepsilon)}}{(K(x_0))^{2/(p+1-\varepsilon)}} \left(1 + O(\varepsilon)\right) \left[1 + O\left(\varepsilon \log(1/\varepsilon)\right)\right] \\ &= \frac{S_n^{(p-1-\varepsilon)/(p+1-\varepsilon)}}{(K(x_0))^{2/(p+1-\varepsilon)}} \left[1 + O\left(\varepsilon \log(1/\varepsilon)\right)\right]. \end{aligned} \tag{3.18}$$

Using (3.18) and Proposition 2.2, (3.17) becomes

$$\begin{aligned} &\frac{S_n^{\frac{p-1-\varepsilon}{p+1-\varepsilon}}}{K(x_\varepsilon)^{\frac{2}{p+1-\varepsilon}}} \left[1 + \frac{c_1 H(x_\varepsilon, x_\varepsilon)}{S_n \lambda_\varepsilon^{n-4}} + \frac{n-4}{n}\varepsilon \left(\log \lambda_\varepsilon^{\frac{n-4}{2}} + \frac{c_3}{S_n} \right) + O\left(\varepsilon^{1+\gamma}\right) \right] \\ &\qquad \le \frac{S_n^{(p-1-\varepsilon)/(p+1-\varepsilon)}}{(K(x_0))^{2/(p+1-\varepsilon)}} \left[1 + O\left(\varepsilon \log(1/\varepsilon)\right)\right]. \end{aligned} \tag{3.19}$$

We now consider two steps.

Step 1. We claim that $x_\varepsilon \notin \{x \mid d(x_\varepsilon, \partial\Omega) = \varepsilon^L\}$. Arguing by contradiction, suppose that $d(x, \partial\Omega) = \varepsilon^L$. Then

$$\lambda_\varepsilon \ge \lambda_{C_0}^\varepsilon(x_\varepsilon) \ge c \left(\frac{1}{d(x_\varepsilon, \partial\Omega)^{n-4}\varepsilon} \right)^{1/(n-4)} \ge \frac{C}{\varepsilon^L}. \tag{3.20}$$

Since $K(x_\varepsilon) \le K(x_0)$ and using (3.20), (3.19) implies

$$\frac{(n-4)^2}{2n}L\varepsilon\log(1/\varepsilon)+O(\varepsilon)\le C\varepsilon\log(1/\varepsilon),$$

where C is a positive constant independent of L. So we get a contradiction if L is chosen large enough.

Step 2. We claim that $x_\varepsilon \notin \partial B_{\varepsilon^\beta}(x_0)$. Again arguing by contradiction we assume that $x_\varepsilon \in \partial B_{\varepsilon^\beta}(x_0)$. Then by assumption on K, we have

$$\frac{1}{K(x_\varepsilon)^{2/(p+1-\varepsilon)}}\ge\frac{1}{\left(K(x_0)-a\varepsilon^{\beta(2+\alpha)}\right)^{2/(p+1-\varepsilon)}}\ge\frac{1+a'\varepsilon^{\beta(2+\alpha)}}{K(x_0)^{2/(p+1-\varepsilon)}},$$

where $a'>0$. Hence, if we can choose $\beta>0$ satisfying

$$\beta(2+\alpha)<1, \tag{3.21}$$

then, using (3.19), we obtain

$$a'\varepsilon^{\beta(2+\alpha)}\le C\varepsilon\log(1/\varepsilon), \tag{3.22}$$

which is impossible. Thus it remains to prove that we can choose a $\beta>0$, such that (3.8) and (3.21) hold. We distinguish two cases: $(i) n\ge 7$ and $(ii) n=5,6$. In the case (i) since $\alpha\in[0,\frac{4}{n-6})$, we can choose $\beta\in(1/2-1/(n-4),1/2)$ satisfying $\beta(2+\alpha)<1$. Finally, if $n=5,6$, we can take $\beta>0$ sufficiently small such that (3.21) holds. From Steps 1 and 2 we deduce that x_ε is an interior point of D_ε.

By construction, the corresponding $u_\varepsilon=P\delta_{x_\varepsilon,\lambda_\varepsilon}+v_\varepsilon$ is a critical point of J_ε, that is, $w_\varepsilon=(l_\varepsilon(u_\varepsilon))^{\frac{1}{p-1-\varepsilon}}u_\varepsilon$ satisfies

$$\Delta^2 w_\varepsilon=K|w_\varepsilon|^{\frac{8}{n-4}-\varepsilon}w_\varepsilon \text{ in } \Omega,\quad w_\varepsilon=\Delta w_\varepsilon=0 \text{ on } \partial\Omega \tag{3.23}$$

with $|w_\varepsilon^-|_{L^{2n/(n-4)}(\Omega)}$ very small, where $w_\varepsilon^-=\max(0,-w_\varepsilon)$. As in Proposition 4.1 of [8], we can prove that $w_\varepsilon^-=0$. Thus, since w_ε is a non-negative function which satisfies (3.23), the strong maximum principle ensures that $w_\varepsilon>0$ on Ω and then u_ε is a solution of (P_ε). This ends the proof of our Theorem. □

Proof of Theorem 1.2 Since the proof of Theorem 1.2 is similar to that of Theorem 1.1, we only point out the necessary changes in the proof. Let $\delta>0$ such that $\forall x\in B_\delta(x_0), K(x)\le K(x_0)$. We consider the minimization problem.

$$\inf\{\psi_\varepsilon(x,\lambda,v_\varepsilon)\mid x\in\overline{B_\delta(x_0)},\lambda\in[\varepsilon^{-\beta},\varepsilon^{-L}]\}, \tag{3.24}$$

in place of (3.3), where $0<\beta<L$ are some constants to be determined later. Let $(x_\varepsilon,\lambda_\varepsilon)$ be a minimizer of problem (3.24). From $\psi_\varepsilon(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)\le\psi_\varepsilon(x_0,\lambda_\varepsilon,0)$, and the fact that x_0 is a strict local maximum, we easily derive that $x_\varepsilon\to x_0$. Next, we show that L and β can be chosen so that $\varepsilon^{-\beta}<\lambda_\varepsilon<\varepsilon^{-L}$. On one hand, it follows from Proposition 2.2 that

$$\psi_\varepsilon(x_0,\varepsilon^{-4},0)=\frac{S_n^{\frac{p-1-\varepsilon}{p+1-\varepsilon}}}{K(x_0)^{\frac{2}{p+1-\varepsilon}}}\left[1+\frac{(n-4)^2}{2n}\varepsilon\left(4\log(1/\varepsilon)+\frac{2c_3}{(n-4)S_n}\right)+O\left(\varepsilon^{1+\sigma}\right)\right]. \tag{3.25}$$

On the other hand, by Lemma 5.2 and (2.17), we have

$$\begin{aligned}
&\int_\Omega K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon} = \int_{B_\delta(x_0)} K(y)|P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon} + O\left(\lambda^{-n}+||v_\varepsilon||^{p+1-\varepsilon}\right)\\
&\le K(x_0)\int_{B_\delta(x_0)} |P\delta_{x,\lambda}+v_\varepsilon|^{p+1-\varepsilon} + O\left(\lambda^{-n}+||v_\varepsilon||^{p+1-\varepsilon}\right)\\
&= K(x_0)\Bigg[\int_\Omega |P\delta_{x,\lambda}|^{p+1-\varepsilon} + (p+1-\varepsilon)\int_\Omega |P\delta_{x,\lambda}|^{p-\varepsilon}v_\varepsilon\\
&+\frac{(p+1-\varepsilon)(p-\varepsilon)}{2}\int_\Omega |P\delta_{x,\lambda}|^{p-1-\varepsilon}v_\varepsilon^2\Bigg] + O\left(\lambda^{-n}+||v_\varepsilon||^{\min(3,p+1-\varepsilon)}\right)\\
&= K(x_0)\Bigg[S_n - (p+1-\varepsilon)\frac{c_1H(x,x)}{\lambda^{n-4}} - \varepsilon S_n\left(\log\lambda^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right) + O\left(\varepsilon+\frac{1}{\lambda^{\frac{n-4}{2}+\theta}}\right)||v_\varepsilon||\\
&+\frac{(p+1-\varepsilon)(p-\varepsilon)}{2}\int_\Omega P\delta_{x,\lambda}^{p-1-\varepsilon}v_\varepsilon^2\Bigg] + O\left(\frac{\varepsilon\log\lambda}{\lambda^{n-4}}+\frac{1}{\lambda^{n-2}}+\varepsilon^2\log^2\lambda+||v_\varepsilon||^{\min(3,p+1-\varepsilon)}\right).
\end{aligned}$$

Clearly, the above estimate implies

$$\begin{aligned}
\psi_\varepsilon(x,\lambda,v_\varepsilon) &\ge \frac{S_n^{\frac{p-1-\varepsilon}{p+1-\varepsilon}}}{K(x_0)^{\frac{2}{p+1-\varepsilon}}}\Bigg[1+\frac{c_1H(x,x)}{S_n\lambda^{n-4}}+\frac{(n-4)}{n}\varepsilon\left(\log\lambda^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right)\\
&+\rho||v_\varepsilon||^2 + O\left(\varepsilon^2\log^2\lambda+\frac{1}{\lambda^{n-4+2\theta}}+\frac{\varepsilon\log\lambda}{\lambda^{n-4}}+\frac{1}{\lambda^{n-2}}\right)\Bigg]\\
&\ge \frac{S_n^{\frac{p-1-\varepsilon}{p+1-\varepsilon}}}{K(x_0)^{\frac{2}{p+1-\varepsilon}}}\left[1+\frac{c_1H(x,x)}{2S_n\lambda^{n-4}}+\frac{(n-4)}{2n}\varepsilon\left(\log\lambda^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right)\right].
\end{aligned}\tag{3.26}$$

Using the inequality $\psi_\varepsilon(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)\le\psi_\varepsilon(x_0,\varepsilon^{-4},0)$, we deduce from (3.25) and (3.26) that

$$\frac{c_1H(x_\varepsilon,x_\varepsilon)}{2S_n\lambda_\varepsilon^{n-4}}+\frac{n-4}{2n}\varepsilon\log\lambda_\varepsilon^{\frac{n-4}{2}} \le \frac{(n-4)^2}{2n}4\varepsilon\log(1/\varepsilon)+O\left(\varepsilon^{1+\sigma}\right). \tag{3.27}$$

As in the proof of Theorem 1.1 we proceed in two steps.
Step 1. We claim that $\lambda_\varepsilon<\varepsilon^{-L}$ for $L>0$ sufficiently large. Arguing by contradiction, suppose that $\lambda_\varepsilon=\varepsilon^{-L}$. Then it follows from (3.27) that

$$\varepsilon L\log(1/\varepsilon)\le 8\varepsilon\log(1/\varepsilon)+O\left(\varepsilon^{1+\sigma}\right),$$

which is impossible if L is large enough.
Step 2. $\lambda_\varepsilon=\varepsilon^{-\beta}$ is impossible if $\beta>0$ is small enough. Assuming that $\lambda_\varepsilon=\varepsilon^{-\beta}$, we deduce from (3.27) that

$$\varepsilon^{(n-4)\beta}\le C\varepsilon\log(1/\varepsilon),$$

which is impossible if β is small enough and therefore our result follows. □

Proof of Theorem 1.5 Arguing by contradiction, suppose that (P_ε) has a solution of the form (1.8) and satisfying (1.9). We start by showing that $\varepsilon\log\lambda_\varepsilon\to 0$ as $\varepsilon\to 0$. Indeed, multiplying

(P_ε) by $P\delta_{x_\varepsilon,\lambda_\varepsilon}$ and integrating over Ω, we obtain

$$\begin{aligned}\alpha_\varepsilon||P\delta_{x_\varepsilon,\lambda_\varepsilon}||^2 &= \int_\Omega K(y)|\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon}+v_\varepsilon|^{p-\varepsilon}P\delta_{x_\varepsilon,\lambda_\varepsilon}\\ &= \alpha_\varepsilon^{p-\varepsilon}\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1-\varepsilon}+O\left(||v_\varepsilon||\right).\end{aligned}$$

As in (2.14), we have

$$\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1-\varepsilon}=\frac{S_nK(x_\varepsilon)}{\lambda_\varepsilon^{\varepsilon(n-4)/2}}(1+o(1)). \tag{3.28}$$

Consequently by (2.13), (3.28) we have

$$\alpha_\varepsilon S_n=\frac{\alpha_\varepsilon^{p-\varepsilon}S_nK(x_\varepsilon)}{\lambda_\varepsilon^{\varepsilon(n-4)/2}}\left(1+o(1)\right)+o(1), \tag{3.29}$$

where $o(1)\to 0$ as $\varepsilon\to 0$. Since $\alpha_\varepsilon\to K(x_0)^{(4-n)/8}$ and $x_\varepsilon\to x_0$ as $\varepsilon\to 0$, we deduce from (3.29) that $\varepsilon\log\lambda_\varepsilon\to 0$ as $\varepsilon\to 0$.

Next, we estimate v_ε. Multiplying (P_ε) by v_ε and integrating over Ω, we obtain

$$\begin{aligned}||v_\varepsilon||^2 &= \int_\Omega K(y)|\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon}+v_\varepsilon|^{p-\varepsilon}v_\varepsilon\\ &= \alpha_\varepsilon^{p-\varepsilon}\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p-\varepsilon}v_\varepsilon+(p-\varepsilon)\alpha_\varepsilon^{p-1-\varepsilon}\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1-\varepsilon}v_\varepsilon^2+O\left(||v_\varepsilon||^{\min(3,p+1-\varepsilon)}\right)\\ &= \alpha_\varepsilon^{p-\varepsilon}\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p-\varepsilon}v_\varepsilon+(p-\varepsilon)\alpha_\varepsilon^{p-1-\varepsilon}K(x_\varepsilon)\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1}v_\varepsilon^2+o\left(||v_\varepsilon||^2\right).\end{aligned} \tag{3.30}$$

It follows from Proposition 3.4 in [5] that there exists a $\rho>0$, such that

$$\begin{aligned}&||v_\varepsilon||^2-(p-\varepsilon)\alpha_\varepsilon^{p-1-\varepsilon}K(x_\varepsilon)\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1}v_\varepsilon^2\\ &\qquad= ||v_\varepsilon||^2-p\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1}v_\varepsilon^2+o\left(||v_\varepsilon||^2\right)\geq\rho||v_\varepsilon||^2.\end{aligned} \tag{3.31}$$

Combining (3.30), (3.31) and with the aid of Lemma 5.2 we get

$$||v_\varepsilon||=O\left(\frac{|DK(x_\varepsilon)|}{\lambda_\varepsilon}+\varepsilon+\frac{1}{\lambda_\varepsilon^2}+\frac{1}{\lambda_\varepsilon^{\theta+(n-4)/2}}\right). \tag{3.32}$$

We now assume that $n\geq 7$. Multiplying (P_ε) by $\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}/\partial\lambda$ and integrating over Ω, we derive that

$$\alpha_\varepsilon\left(P\delta_{x_\varepsilon,\lambda_\varepsilon},\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda}\right)-\alpha_\varepsilon^{(p-\varepsilon)}\int_\Omega K(y)|P\delta_{x_\varepsilon,\lambda_\varepsilon}+v_\varepsilon|^{p-\varepsilon}\frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda}=0.$$

Arguing as in the proof of Lemma 2.4, we easily arrive at

$$\frac{c_2\Delta K(x_\varepsilon)}{n\lambda_\varepsilon^3}+\frac{(n-4)S_nK(x_\varepsilon)\varepsilon}{2\lambda_\varepsilon}+O\left(\frac{1}{\lambda_\varepsilon^4}+\frac{\varepsilon\log\lambda_\varepsilon}{\lambda_\varepsilon^3}+\frac{\varepsilon^2\log\lambda_\varepsilon}{\lambda_\varepsilon}+\frac{|DK(x_\varepsilon)|^2}{\lambda_\varepsilon^3}+\frac{\varepsilon^2}{\lambda_\varepsilon}\right)=0. \tag{3.33}$$

Since $\Delta K(x_\varepsilon) > 0$ and $\varepsilon \log \lambda_\varepsilon \to 0$ as $\varepsilon \to 0$, we get from (3.33) that

$$\frac{1}{\lambda_\varepsilon^3} + \frac{\varepsilon}{\lambda_\varepsilon} \leq 0,$$

which is impossible.
Finally, we consider the case $n = 6$. As in the case $n \geq 7$ we derive the following relation

$$\left(\frac{c_2 \Delta K(x_\varepsilon)}{36 K(x_\varepsilon)} - c_1 H(x_\varepsilon, x_\varepsilon)\right) \frac{1}{\lambda_\varepsilon^3} + \frac{S_n \varepsilon}{6\lambda_\varepsilon} + o\left(\frac{\varepsilon}{\lambda_\varepsilon} + \frac{1}{\lambda_\varepsilon^3}\right) = 0,$$

which contradicts the assumption (ii). This completes the proof of Theorem 1.5. □

4 Proof of Theorems 1.3, 1.4 and 1.6

In this section, except in the proof of Theorems 1.4 and 1.6, we always assume that K satisfies the conditions in Theorem 1.3. We now start by proving the following propositions.

Proposition 4.1 *There exists an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a C^1-map $\lambda_\varepsilon : B_{\varepsilon^{1/L}} \to \mathbb{R}^+$, $x \mapsto \lambda_\varepsilon(x)$, such that $\lambda_\varepsilon(x)$ satisfies (E_λ). Moreover, $\lambda_\varepsilon(x) = t_\varepsilon(x)\varepsilon^{-1/(n-4)}$ with*

$$|t_\varepsilon(x) - t_0(x)| = O\left(\varepsilon^\sigma\right), \tag{4.1}$$

where $\sigma > 0$ and

$$t_0(x) = \left(\frac{2nc_1 H(x,x)}{(n-4)S_n}\right)^{1/(n-4)}.$$

Proof. Using Lemma 2.4, and the fact that (2.25) holds, we obtain

$$\frac{\partial \psi_\varepsilon}{\partial \lambda}(x, \lambda, v_\varepsilon) = \frac{1}{(S_n K(x))^{\frac{2}{p+1-\varepsilon}}} \left[\frac{-c_1(n-4)H(x,x)}{\lambda^{n-3}} + \frac{(n-4)^2 S_n \varepsilon}{2n\lambda} + O\left(\varepsilon^{1+\frac{1}{n-4}+\sigma}\right)\right]. \tag{4.2}$$

On the other hand by Lemmas 2.5, 2.11 we have

$$B\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial \lambda^2}, v_\varepsilon\right) + \sum_{j=1}^n C_j \left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial x_j \partial \lambda}, v_\varepsilon\right) = O\left(\varepsilon^{1+\frac{1}{n-4}+\frac{1-\sigma}{2}}\right). \tag{4.3}$$

Consequently, equation (E_λ) is equivalent to

$$-\frac{c_1(n-4)H(x,x)}{\lambda^{n-3}} + \frac{(n-4)^2 S_n \varepsilon}{2n\lambda} + O\left(\varepsilon^{1+\frac{1}{n-4}+\sigma}\right) = 0. \tag{4.4}$$

Letting $\lambda_\varepsilon = t_\varepsilon \varepsilon^{-1/(n-4)}$, we deduce from (4.4) that

$$-\frac{c_1 H(x,x)}{t_\varepsilon^{n-3}} + \frac{(n-4)S_n}{2nt_\varepsilon} + O\left(\varepsilon^\sigma\right) = 0. \tag{4.5}$$

It is easy to see that (4.5) has a solution

$$t_\varepsilon \in \left(\frac{1}{2}\left(\frac{2nc_1 H(x,x)}{(n-4)S_n}\right)^{\frac{1}{n-4}}, \frac{3}{2}\left(\frac{2nc_1 H(x,x)}{(n-4)S_n}\right)^{\frac{1}{n-4}} \right).$$

This implies the existence of $\lambda_\varepsilon(x)$ satisfying (E_λ). Next, we show that $\lambda_\varepsilon(x)$ is a C^1-map in x. To this aim, let

$$F(\lambda) = \frac{\partial \psi_\varepsilon}{\partial \lambda} - B\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial \lambda^2}, v_\varepsilon\right) - \sum_{j=1}^{n} C_j \left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial x_j \partial \lambda}, v_\varepsilon\right).$$

Then it follows from Lemma 2.10, Lemma 2.11 and Proposition 2.9 that

$$F'(\lambda) = \frac{1}{(K(x)S_n)^{\frac{2}{p+1-\varepsilon}}} \left(\frac{(n-3)(n-4)c_1 H(x,x)}{\lambda^{n-2}} - \frac{(n-4)^2 S_n \varepsilon}{2n\lambda^2} \right) + O\left(\varepsilon^{1+\frac{2}{n-4}+\sigma}\right) > 0, \quad (4.6)$$

for all $\lambda \in \left(\frac{1}{2}\left(\frac{2nc_1 H(x,x)}{(n-4)S_n\varepsilon}\right)^{\frac{1}{n-4}}, \frac{3}{2}\left(\frac{2nc_1 H(x,x)}{(n-4)S_n\varepsilon}\right)^{\frac{1}{n-4}} \right)$. Consequently, the equation (E_λ) has a unique solution in

$$\lambda \in \left(\frac{1}{2}\left(\frac{2nc_1 H(x,x)}{(n-4)S_n\varepsilon}\right)^{\frac{1}{n-4}}, \frac{3}{2}\left(\frac{2nc_1 H(x,x)}{(n-4)S_n\varepsilon}\right)^{\frac{1}{n-4}} \right),$$

and since all the terms in (E_λ) are of C^1 with respect to x and λ, we deduce that $\lambda_\varepsilon(x)$ is a C^1 map in x.

Now, let

$$\Phi(t) = -\frac{(n-4)c_1 H(x,x)}{t^{n-3}} + \frac{(n-4)^2 S_n}{2nt}.$$

We then have

$$\Phi(t_0(x)) = 0, \quad \Phi(t_\varepsilon(x)) = O\left(\varepsilon^\sigma\right). \quad (4.7)$$

Since $\Phi'(t_0(x)) > 0$, it follows from (4.7) that $|t_\varepsilon(x) - t_0(x)| = O\left(\varepsilon^\sigma\right)$ and this completes the proof of Proposition 4.1. $\square$

Now, we consider the following maximization problem

$$\sup\{\psi_\varepsilon(x, \lambda_\varepsilon(x), v_\varepsilon(x, \lambda_\varepsilon(x))) \mid |x - x_0| \le \varepsilon^{1/L}\}. \quad (4.8)$$

Then (4.8) has a maximizer $x_\varepsilon \in \{x \mid |x - x_0| \le \varepsilon^{1/L}\}$. In order to prove that x_ε is a critical point, we only need to prove that $|x_\varepsilon - x_0| < \varepsilon^{1/L}$.

Proposition 4.2 *Let x_ε be a maximizer of* (4.8). *Then there exists a $\sigma_2 > 0$, such that $|x_\varepsilon - x_0|^L = O\left(\varepsilon^{1+\sigma_2}\right)$. In particular, if $\varepsilon > 0$ is small enough, x_ε is an interior point of $B_{\varepsilon^{1/L}}(x_0)$.*

Proof. It follows from Lemma 2.5, Propositions 2.2, and 4.1 that

$$\begin{aligned}\psi_\varepsilon(x,&\lambda_\varepsilon(x), v_\varepsilon(x,\lambda_\varepsilon(x)))\\ &= \frac{S_n^{\frac{p-1-\varepsilon}{p+1+\varepsilon}}}{K(x)^{\frac{2}{p+1-\varepsilon}}}\left[1+\frac{c_1 H(x,x)}{S_n\lambda_\varepsilon^{n-4}}+\frac{(n-4)\varepsilon}{n}\left(\log\lambda_\varepsilon^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right)+O\left(\varepsilon^{1+\sigma}\right)\right]. \qquad (4.9)\end{aligned}$$

Letting

$$\lambda_\varepsilon(x) := t_\varepsilon(x)\varepsilon^{-1/(n-4)} = \left(t_0(x)+O(\varepsilon^\sigma)\right)\varepsilon^{-1/(n-4)},$$

we deduce from (4.9) that

$$\begin{aligned}\psi_\varepsilon(x,\lambda_\varepsilon(x), v_\varepsilon(x,\lambda_\varepsilon(x))) = \frac{S_n^{\frac{p-1-\varepsilon}{p+1+\varepsilon}}}{K(x)^{\frac{2}{p+1-\varepsilon}}}\Bigg[&1+\frac{c_1 H(x,x)\varepsilon}{S_n t_0(x)^{n-4}}+\frac{(n-4)^2\varepsilon}{2n}\log t_0(x)\\ &+\frac{n-4}{2n}\varepsilon\log(1/\varepsilon)+\frac{(n-4)c_3\varepsilon}{nS_n}+O\left(\varepsilon^{1+\sigma}\right)\Bigg]. \qquad (4.10)\end{aligned}$$

Since x_ε is a maximum of (4.8), we have

$$\psi_\varepsilon(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon), v_\varepsilon(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon))) \geq \psi_\varepsilon(x_0,\lambda_\varepsilon(x_0), v_\varepsilon(x_0,\lambda_\varepsilon(x_0))).$$

This, together with (4.10) and the assumption

$$K(x_\varepsilon) \geq K(x_0) + C_0|x_\varepsilon - x_0|^L,$$

imply

$$\begin{aligned}|x_\varepsilon - x_0|^L &\leq C\varepsilon\left(\log H(x_\varepsilon,x_\varepsilon) - \log H(x_0,x_0)\right) + O\left(\varepsilon^{1+\sigma}\right)\\ &= O\left(\varepsilon|x_\varepsilon - x_0|\right) + O\left(\varepsilon^{1+\sigma}\right).\end{aligned}$$

Hence $|x_\varepsilon - x_0|^L = O\left(\varepsilon^{1+\sigma_2}\right)$, where σ_2 is a positive constant. Thus Proposition 4.2 follows. □

Proof of Theorem 1.3 We only need to prove that $(x_\varepsilon, \lambda_\varepsilon(x_\varepsilon), v_\varepsilon(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon)))$ satisfies (E_x). Indeed, we have by easy computations

$$\begin{aligned}0 &= \frac{\partial\psi_\varepsilon}{\partial x_i}+\frac{\partial\psi_\varepsilon}{\partial\lambda}\frac{\partial\lambda}{\partial x_i}+\left(\frac{\partial\psi_\varepsilon}{\partial v},\frac{\partial v}{\partial x_i}+\frac{\partial v}{\partial\lambda}\frac{\partial\lambda}{\partial x_i}\right)\\ &= \frac{\partial\psi_\varepsilon}{\partial x_i}+\left[B\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda^2},v\right)+\sum_{j=1}^n C_j\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda\partial x_j},v\right)\right]\frac{\partial\lambda}{\partial x_i}+B\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},\frac{\partial v}{\partial x_i}\right)\\ &+\sum_{j=1}^n C_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j},\frac{\partial v}{\partial x_i}\right)+\left[B\left(\frac{\partial P\delta_{x,\lambda}}{\partial\lambda},\frac{\partial v}{\partial\lambda}\right)+\sum_{j=1}^n C_j\left(\frac{\partial P\delta_{x,\lambda}}{\partial x_j},\frac{\partial v}{\partial\lambda}\right)\right]\frac{\partial\lambda}{\partial x_i}\\ &= \frac{\partial\psi_\varepsilon}{\partial x_i}-B\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial\lambda\partial x_i},v\right)-\sum_{j=1}^n C_j\left(\frac{\partial^2 P\delta_{x,\lambda}}{\partial x_i\partial x_j},v\right).\end{aligned}$$

This obviously shows that (E_x) holds and as in the proof of Theorem 1.1, we see that the corresponding $u_\varepsilon = P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ is a solution of (P_ε). $\square$

Proof of Theorem 1.4 Theorem 1.4 can be proved in exactly the same way as Theorem 1.3 $\square$

Proof of Theorem 1.6 Arguing by contradiction, let us suppose that (Q_ε) has a solution of the form (1.8) and satisfying (1.9). We start by showing that λ_ε occurring in (1.8) satisfies $\lambda_\varepsilon^{\varepsilon(n-4)/2} \to 1$ as $\varepsilon \to 0$. Indeed, multiplying $(Q_{\varepsilon)}$ by $P\delta_{x_\varepsilon,\lambda_\varepsilon}$ and integrating over Ω, we obtain

$$\begin{aligned}
\alpha_\varepsilon ||P\delta_{x_\varepsilon,\lambda_\varepsilon}||^2 &= \int_\Omega K(y)|\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^{p+\varepsilon} P\delta_{x_\varepsilon,\lambda_\varepsilon} \\
&= \alpha_\varepsilon^{p+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1+\varepsilon} + O\left(\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p+\varepsilon}|v_\varepsilon| + \int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}|v_\varepsilon|^{p+\varepsilon}\right) \\
&= \alpha_\varepsilon^{p+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1+\varepsilon} + O\left(\lambda_\varepsilon^{\varepsilon(n-4)/2}\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p}|v_\varepsilon| + \lambda_\varepsilon^{\varepsilon(n-4)/2}\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{1-\varepsilon}|v_\varepsilon|^{p+\varepsilon}\right) \\
&= \alpha_\varepsilon^{p+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1+\varepsilon} + O\left(\lambda_\varepsilon^{\varepsilon(n-4)/2}||v|| + \lambda_\varepsilon^{\varepsilon(n-4)/2}||v||^{p+\varepsilon}\right).
\end{aligned}$$

As in (2.14), we have

$$\begin{aligned}
\int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1+\varepsilon} &= \int_\Omega K(y)\left(\delta_{x_\varepsilon,\lambda_\varepsilon} - \varphi_{x_\varepsilon,\lambda_\varepsilon}\right)^{p+1+\varepsilon} \\
&= \int_\Omega K(y)\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+1+\varepsilon} + O\left(\int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p+\varepsilon}\varphi_{x_\varepsilon,\lambda_\varepsilon}\right) \\
&= K(x_\varepsilon)c_0^\varepsilon \lambda^{\varepsilon(n-4)/2}\int_{\mathbb{R}^n}\delta_{0,1}^{p+1+\varepsilon} + o(\lambda^{\varepsilon(n-4)/2}) \\
&= S_n K(x_\varepsilon)\lambda_\varepsilon^{\varepsilon(n-4)/2}(1+o(1)). \qquad (4.11)
\end{aligned}$$

Consequently by (2.13) and (4.11), we have

$$\alpha_\varepsilon S_n = \alpha_\varepsilon^{p+\varepsilon} S_n K(x_\varepsilon)\lambda_\varepsilon^{\varepsilon(n-4)/2}\left(1+o(1)\right) + o(1). \qquad (4.12)$$

Since $\alpha_\varepsilon \to K(x_0)^{(4-n)/8}$ and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, we deduce from (4.12) that $\lambda_\varepsilon^{\varepsilon(n-4)/2} \to 1$ as $\varepsilon \to 0$.

Next, we are going to estimate the v_ε-part of u_ε. Multiplying $(Q_{\varepsilon)}$ by v_ε and integrating over Ω, we obtain

$$\begin{aligned}
||v_\varepsilon||^2 &= \int_\Omega K(y)|\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^{p+\varepsilon} v_\varepsilon \\
&= \alpha_\varepsilon^{p+\varepsilon}\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+\varepsilon}v_\varepsilon + (p+\varepsilon)\alpha_\varepsilon^{p-1+\varepsilon}\int_\Omega K(y)P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1+\varepsilon}v_\varepsilon^2 \\
&\quad + O\left(||v_\varepsilon||^3 + \int_\Omega |v_\varepsilon|^{p+1+\varepsilon}\right). \qquad (4.13)
\end{aligned}$$

According to Lemmas 4.4 and 4.5 of [6], we have

$$\int_\Omega |v_\varepsilon|^{p+1+\varepsilon} = o(1) \quad \text{and} \quad |v|^\varepsilon_{L^\infty(\Omega)} = O(1),$$

therefore

$$||v_\varepsilon||^2 - (p+\varepsilon)\alpha_\varepsilon^{p-1+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1+\varepsilon} v_\varepsilon^2 = \alpha_\varepsilon^{p+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+\varepsilon} v_\varepsilon + +O(||v_\varepsilon||^{inf(3,p+1)}).$$

Observe that

$$\begin{aligned} ||v_\varepsilon||^2 - (p+\varepsilon)\alpha_\varepsilon^{p-1+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1+\varepsilon} v_\varepsilon^2 &= ||v_\varepsilon||^2 - (p+\varepsilon)\alpha_\varepsilon^{p-1+\varepsilon} K(x_\varepsilon) \int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1+\varepsilon} v_\varepsilon^2 + o(||v||^2) \\ &= ||v_\varepsilon||^2 - (p+\varepsilon)\alpha_\varepsilon^{p-1+\varepsilon} K(x_\varepsilon) c_0^\varepsilon \lambda^{\varepsilon(n-4)/2} \int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1} v_\varepsilon^2 + o(||v||^2). \end{aligned} \tag{4.14}$$

Since $x_\varepsilon \to x_0$, $\alpha_\varepsilon \to K(x_0)^{(4-n)/8}$ and $\lambda^{\varepsilon(n-4)/2} \to 1$ as $\varepsilon \to 0$, it follows from Proposition 3.4 of [5] that

$$||v_\varepsilon||^2 - (p+\varepsilon)\alpha_\varepsilon^{p-1+\varepsilon} K(x_\varepsilon) c_0^\varepsilon \lambda^{\varepsilon(n-4)/2} \int_\Omega \delta_{x_\varepsilon,\lambda_\varepsilon}^{p-1} v_\varepsilon^2 \geq \rho ||v_\varepsilon||^2, \tag{4.15}$$

where ρ is a positive constant independent of ε.
As in Lemma 5.2, we have

$$\alpha_\varepsilon^{p+\varepsilon} \int_\Omega K(y) P\delta_{x_\varepsilon,\lambda_\varepsilon}^{p+\varepsilon} v_\varepsilon = O\left(\frac{|DK(x_\varepsilon)|}{\lambda_\varepsilon} + \varepsilon + \frac{1}{\lambda_\varepsilon^2} + \frac{1}{\lambda_\varepsilon^{\theta+(n-4)/2}} \right) \tag{4.16}$$

Then we deduce from (4.13) and (4.16) that

$$||v_\varepsilon|| = O\left(\frac{|DK(x_\varepsilon)|}{\lambda_\varepsilon} + \varepsilon + \frac{1}{\lambda_\varepsilon^2} + \frac{1}{\lambda_\varepsilon^{\theta+(n-4)/2}} \right). \tag{4.17}$$

Now, multiplying (Q_ε) by $\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}/\partial\lambda$ and integrating over Ω, we derive that

$$\alpha_\varepsilon \left(P\delta_{x_\varepsilon,\lambda_\varepsilon}, \frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial \lambda} \right) - \int_\Omega K(y) |\alpha_\varepsilon P\delta_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^{p+\varepsilon} \frac{\partial P\delta_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda} = 0.$$

Arguing as in the proof of Lemma 2.4, we easily arrive at

$$\begin{aligned} -\frac{c_2 \Delta K(x_\varepsilon)}{n^2 K(x_\varepsilon)\lambda_\varepsilon^3} + \frac{c_1 H(x_\varepsilon, x_\varepsilon)}{\lambda_\varepsilon^{n-3}} + \frac{(n-4)S_n\varepsilon}{2n\lambda_\varepsilon} + O\left(\frac{1}{\lambda_\varepsilon^4} + \frac{\varepsilon \log \lambda_\varepsilon}{\lambda_\varepsilon^3} + \frac{\varepsilon^2 \log\lambda_\varepsilon}{\lambda_\varepsilon} \right) \\ + \left(\frac{|DK(x_\varepsilon)|^2}{\lambda_\varepsilon^3} + \frac{\varepsilon^2}{\lambda} + \frac{1}{\lambda_\varepsilon^{n-3+2\theta}} + \frac{\varepsilon \log\lambda_\varepsilon}{\lambda_\varepsilon^{n-3}} \right) = 0. \end{aligned} \tag{4.18}$$

For $n = 5$, it follows from (4.18) that

$$\frac{c_1 H(x_\varepsilon, x_\varepsilon)}{\lambda_\varepsilon^2} + \frac{(n-4)S_n\varepsilon}{2n\lambda_\varepsilon} + o\left(\frac{1}{\lambda_\varepsilon^2} + \frac{\varepsilon}{\lambda_\varepsilon} \right) = 0,$$

which is impossible.
For $n=6$, we derive form (4.18) the following relation

$$\left(-\frac{c_2\Delta K(x_\varepsilon)}{36K(x_\varepsilon)}+c_1H(x_\varepsilon,x_\varepsilon)\right)\frac{1}{\lambda_\varepsilon^3}+\frac{S_n\varepsilon}{6\lambda_\varepsilon}+o\left(\frac{1}{\lambda_\varepsilon^3}+\frac{\varepsilon}{\lambda_\varepsilon}\right)=0,$$

which is a contradiction with the assumption (ii).
Finally, for $n\geq 7$, we derive the following relation

$$-\frac{c_2\Delta K(x_\varepsilon)}{n^2K(x_\varepsilon)\lambda_\varepsilon^3}+\frac{(n-4)S_n\varepsilon}{2n\lambda_\varepsilon}+o\left(\frac{1}{\lambda_\varepsilon^3}+\frac{\varepsilon}{\lambda_\varepsilon}\right)=0,$$

which contradicts the assumption (iii). This completes the proof of Theorem 1.6. □

5 Appendix

In this appendix, we collect the integral estimates which are needed in Section 2.

Lemma 5.1 *Suppose that $\lambda d(x,\partial\Omega)\to+\infty$, $\varepsilon\log\lambda\to 0$ as $\varepsilon\to 0$.*
Then the following estimates hold

1. $$\int_\Omega K(y)\delta_{x,\lambda}^{p+1-\varepsilon}(y)dy=K(x)S_n+\frac{c_2\Delta K(x)}{2n\lambda^2}-\varepsilon K(x)S_n\left(\log\lambda^{\frac{n-4}{2}}+\frac{c_3}{S_n}\right)$$
$$+O\left(\sum_{j=3}^{n-4}\frac{|D^jK(x)|}{\lambda^j}+\frac{1}{\lambda^{n-3}}+\frac{\varepsilon\log\lambda}{\lambda^2}+\varepsilon^2\log^2\lambda+\frac{1}{(\lambda d)^n}\right),$$

   where $S_n=\int_{\mathbb{R}^n}\delta_{o,1}^{p+1}$, $c_2=\int_{\mathbb{R}^n}|y|^2\delta_{o,1}^{p+1}dy$, $c_3=\int_{\mathbb{R}^n}\delta_{o,1}^{p+1}\log\delta_{o,1}(y)dy$ *and* $d=d(x,\partial\Omega)$.

2. $$\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\varphi_{x,\lambda}=\frac{c_1K(x)H(x,x)}{\lambda^{n-4}}+O\left(\frac{\varepsilon\log\lambda}{(\lambda d)^{n-4}}+\frac{1}{(\lambda d)^{n-2}}\right),$$

   where $c_1=c_0^{\frac{2n}{n-4}}\int_{\mathbb{R}^n}\frac{dy}{(1+|y|^2)^{(n+4)/2}}$ *and* c_0 *is defined in* (1.1).

3. $$\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}(y)\frac{\partial\delta_{x,\lambda}}{\partial\lambda}dy=-\frac{K(x)(n-4)^2S_n\varepsilon}{4n\lambda}-\frac{(n-4)}{2n^2}c_2\frac{\Delta K(x)}{\lambda^3}$$
$$+O\left(\frac{\varepsilon^2\log\lambda}{\lambda}+\frac{\varepsilon\log\lambda}{\lambda^3}+\sum_{j=3}^{n-4}\frac{|D^jK(x)|}{\lambda^{j+1}}+\frac{1}{\lambda^{n-2}}+\frac{1}{\lambda(\lambda d)^n}\right).$$

4. $$\int_\Omega K(y)\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}\frac{\partial\delta_{x,\lambda}}{\partial\lambda}=-\frac{(n-4)^2}{2(n+4)}\frac{c_1K(x)H(x,x)}{\lambda^{n-3}}+O\left(\frac{\varepsilon\log\lambda}{\lambda(\lambda d)^{n-4}}+\frac{1}{\lambda(\lambda d)^{n-2}}\right).$$

5. $$\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}=-\frac{(n-4)}{2}\frac{c_1K(x)H(x,x)}{\lambda^{n-3}}+O\left(\frac{\varepsilon\log\lambda}{\lambda(\lambda d)^{n-4}}+\frac{1}{\lambda(\lambda d)^{n-2}}\right).$$

Proof. Using the fact that $\delta_{x,\lambda}^{-\varepsilon} = 1 - \varepsilon \log \delta_{x,\lambda} + O\left(\varepsilon^2 \log^2 \lambda\right)$, we obtain

$$\begin{aligned}
\int_\Omega K(y)\delta_{x,\lambda}^{p+1-\varepsilon}(y)dy &= \int_\Omega K(y)\delta_{x,\lambda}^{p+1}(1-\varepsilon\log\delta_{x,\lambda}) + O\left(\int_\Omega \delta_{x,\lambda}^{p+1}(\varepsilon\log\lambda)^2\right)\\
&= \int_{B(x,d)} K(y)\delta_{x,\lambda}^{p+1} - \varepsilon\log\lambda^{\frac{n-4}{2}}\int_{B(x,d)} K(y)\delta_{x,\lambda}^{p+1}\\
&- \varepsilon\int_{B(x,d)} K(y)\delta_{x,\lambda}^{p+1}\log\left(\frac{c_0}{(1+\lambda^2|y-x|^2)^{(n-4)/2}}\right)\\
&+ O\left(\varepsilon^2\log^2\lambda + (\lambda d)^{-n}\right).
\end{aligned}$$

Thus, using Taylor's expansion, we easily derive Claim 1.
Now, using (see [9])

$$\varphi_{x,\lambda} = c_0\frac{H(x,y)}{\lambda^{(n-4)/2}} + O\left(\frac{1}{\lambda^{n/2}d^{n-2}}\right),$$

we derive that

$$\begin{aligned}
\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\varphi_{x,\lambda} &= \int_\Omega \delta_{x,\lambda}^{p-\varepsilon}\frac{c_0K(y)H(x,y)}{\lambda^{(n-4)/2}} + O\left(\frac{1}{(\lambda d)^{n-2}}\right)\\
&= \int_{B(x,d)} \delta_{x,\lambda}^{p}\frac{c_0K(y)H(x,y)}{\lambda^{(n-4)/2}} + O\left(\frac{\varepsilon\log\lambda}{(\lambda d)^{n-4}} + \frac{1}{(\lambda d)^{n-2}}\right)\\
&= \frac{c_0K(x)H(x,x)}{\lambda^{(n-4)/2}}\int_{\mathbb{R}^n}\delta_{x,\lambda}^{p} + O\left(\frac{\varepsilon\log\lambda}{(\lambda d)^{n-4}} + \frac{1}{(\lambda d)^{n-2}}\right)
\end{aligned}$$

and therefore Claim 2 follows.
To prove Claim 3, we use again Taylor's expansion and we thus obtain

$$\begin{aligned}
\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial\delta_{x,\lambda}}{\partial\lambda} &= K(x)\int_{\mathbb{R}^n}\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial\delta_{x,\lambda}}{\partial\lambda} + \frac{\Delta K(x)}{2n}\int_{\mathbb{R}^n}\delta_{x,\lambda}^{p-\varepsilon}|y-x|^2\frac{\partial\delta_{x,\lambda}}{\partial\lambda}\\
&+ O\left(\sum_{j=3}^{n-4}\frac{|D^jK(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^{n-2}} + \frac{1}{\lambda(\lambda d)^n}\right)\\
&= \frac{K(x)}{p+1-\varepsilon}\frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda^{\varepsilon\frac{n-4}{2}}}\int_{\mathbb{R}^n}\delta_{o,1}^{p+1-\varepsilon}\right)\\
&+ \frac{\Delta K(x)}{2n(p+1-\varepsilon)}\frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda^{2+\varepsilon\frac{n-4}{2}}}\int_{\mathbb{R}^n}\delta_{o,1}^{p+1-\varepsilon}|y|^2\right)\\
&+ O\left(\sum_{j=3}^{n-4}\frac{|D^jK(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^{n-2}} + \frac{1}{\lambda(\lambda d)^n}\right)\\
&= -K(x)\varepsilon\frac{(n-4)^2S_n}{4n\lambda}(1+O(\varepsilon))(1+O(\varepsilon\log\lambda))
\end{aligned}$$

$$- c_2 \frac{\Delta K(x)(n-4)}{2n^2\lambda^3}(1+O(\varepsilon))(1+O(\varepsilon\log\lambda))$$
$$+ O\left(\sum_{j=3}^{n-4}\frac{|D^jK(x)|}{\lambda^{j+1}}+\frac{1}{\lambda^{n-2}}+\frac{1}{\lambda(\lambda d)^n}\right).$$

Thus Claim 3 follows.
Now we are going to prove Claim 4. To this aim, we write

$$\int_\Omega K(y)\delta_{x,\lambda}^{p-1-\varepsilon}\varphi_{x,\lambda}\frac{\partial\delta_{x,\lambda}}{\partial\lambda} = \int_\Omega K(y)\delta_{x,\lambda}^{p-1}\varphi_{x,\lambda}\frac{\partial\delta_{x,\lambda}}{\partial\lambda} + O\left(\frac{\varepsilon\log\lambda}{\lambda}\int_\Omega \delta_{x,\lambda}^p\varphi_{x,\lambda}\right)$$
$$= c_0\frac{K(x)H(x,x)}{\lambda^{\frac{n-4}{2}}}\int_{B(x,d)}\delta_{x,\lambda}^{p-1}\frac{\partial\delta_{x,\lambda}}{\partial\lambda} + O\left(\frac{1}{\lambda(\lambda d)^{n-2}}+\frac{\varepsilon\log\lambda}{\lambda(\lambda d)^{n-4}}\right)$$

and we can thus easily derive Claim 4.
Lastly, we have

$$\int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda} = \int_\Omega K(y)\delta_{x,\lambda}^{p}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda} + O\left(\frac{\varepsilon\log\lambda}{\lambda(\lambda d^2)^{\frac{n-4}{2}}}\int_\Omega\delta_{x,\lambda}^p\right)$$
$$= -c_0\frac{n-4}{2}\int_\Omega\frac{K(y)H(x,y)}{\lambda^{\frac{n-2}{2}}}\delta_{x,\lambda}^p + O\left(\frac{1}{\lambda^{\frac{n+2}{2}}d^{n-2}}\int_\Omega\delta_{x,\lambda}^p+\frac{\varepsilon\log\lambda}{\lambda(\lambda d)^{n-4}}\right)$$

and thus Claim 5 follows. The proof of Lemma 5.1 is thereby completed. □

Lemma 5.2 *Let k be the biggest positive integer satisfying $k \le (n-4)/2$. Thus, for any $v \in E_{x,\lambda}$, we have*

1. $$\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}v = O\left(\varepsilon+\sum_{j=1}^{k}\frac{|D^jK(x)|}{\lambda^j}+\frac{1}{\lambda^{k+1}}+\frac{1}{(\lambda d)^{\frac{n-4}{2}+\theta}}\right)||v||,$$

2. $$\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}v\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} = O\left(\frac{\varepsilon}{\lambda}+\sum_{j=1}^{k}\frac{|D^jK(x)|}{\lambda^{j+1}}+\frac{1}{\lambda^{k+2}}+\frac{1}{\lambda(\lambda d)^{\frac{n-4}{2}+\theta}}\right)||v||,$$

3. $$\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}v\frac{\partial P\delta_{x,\lambda}}{\partial x} = O\left(\varepsilon\lambda+\sum_{j=1}^{k}\frac{|D^jK(x)|}{\lambda^{j-1}}+\frac{1}{\lambda^{k}}+\frac{\lambda}{(\lambda d)^{\frac{n-4}{2}+\theta}}\right)||v||,$$

where θ is a positive constant.

Proof. We observe that

$$\int_\Omega K(y)P\delta_{x,\lambda}^{p-\varepsilon}v = \int_\Omega K(y)\delta_{x,\lambda}^{p-\varepsilon}v + O\left(\int_\Omega\delta_{x,\lambda}^{p-1-\varepsilon}|\varphi_{x,\lambda}||v|\right)$$
$$= K(x)\int_\Omega\delta_{x,\lambda}^p(1-\varepsilon\log(c_0\lambda^{(n-4)/2})v + O\left(\sum_{j=1}^{k}\frac{|D^jK(x)|}{\lambda^j}+\frac{1}{\lambda^{k+1}}\right)||v||$$
$$+ O\left(\varepsilon\int_\Omega\delta_{x,\lambda}^p\log(1+\lambda^2|x-a|^2)|v|\right)+O\left(\frac{||v||}{(\lambda d)^{\frac{n-4}{2}+\theta}}\right)$$

and thus Claim 1 follows.
We also observe that

$$\int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}v\frac{\partial P\delta_{x,\lambda}}{\partial\lambda} = \int_\Omega K(y)P\delta_{x,\lambda}^{p-1-\varepsilon}v\frac{\partial\delta_{x,\lambda}}{\partial\lambda} + O\left(\int_\Omega \delta_{x,\lambda}^{p-1-\varepsilon}|v||\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}|\right)$$
$$= \int_\Omega K(y)\delta_{x,\lambda}^{p-1-\varepsilon}v\frac{\partial\delta_{x,\lambda}}{\partial\lambda} + O\left(\frac{1}{\lambda}\int_\Omega \delta_{x,\lambda}^{p-1-\varepsilon}|v|\varphi_{x,\lambda}\right) + O\left(\frac{||v||}{\lambda(\lambda d)^{\frac{n-4}{2}+\theta}}\right).$$

Thus, using Taylor's expansion, we easily derive Claim 2.
In the same way, we can prove Claim 3 and therefore the proof of our lemma is completed. □

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